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Optimum Design of Shapes Using the Pontryagin Principle of Maximum

1. Introduction

Optimum shape design is an interesting and important field both mathematically and for industrial applications. Uniqueness, stability and existence of solution are important theoretical issues. Practical implementation issues are critical for realization. As an example of industrial applications we can consider weight reduction in car engine, aircraft structures, drag reduction for airplanes, cars and boats, electromagnetically optimum shapes, such as in stealth airplanes, etc.

In this paper, we investigate a methodology for the shape optimization problem. The problem consists in finding a shape (in two or three dimensions), which is optimum in a certain sense and satisfies certain requirements. In other words, we would like to find a bounded set D, which minimizes a functional $J(D)$ and satisfies constraints $B(D) = 0$. The method based on the Pontryagin maximum principle (Pontryagin *et al.* 1962; Boltyanskii 1971; Mitkowski 1991) can be used for solving the formulated task of optimization. Of course, a general solution and proof are very often impossible. Therefore the main focus in this paper will be put on numerical solutions. The computer program has been designed in MATLAB/Simulink™ environment. It uses an iterative method, that is, we start with an initial guess for a shape, and then gradually evolve it, until it falls into the optimum shape. Using this program we show how to find optimum shapes for different types of beam designs.

Let us consider a physical system of which the state equation is of the form

$$\frac{dx}{d\xi} = f(x, u, \xi) \tag{1}$$

where $x(\xi) \in \mathbb{R}^n$, $u(\xi) \in U_{ad} \subset \mathbb{R}$, $\xi \; \mathbb{R} \; [\xi_0, \xi_1]$, f is a vector function that is differentiable in x and ξ with continuous derivatives.

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The equation (1) with the boundary conditions

$$r(x(\xi_0), x(\xi_1)) = 0 \tag{2}$$

can describe statics of the system. The function r is such that for every control function u the boundary value problem (1), (2) has a unique solution x. In such situation the vector x represents geometrical variables along the interval $[\xi_0, \xi_1]$, U_{ad} stands for the set of admissible controls and it is usually determined by geometrical and strength constraints. Existence of the constraints secures the solutions of (1) the proper physical meaning. It should be noted that not all elements of vectors $x(\xi_0)$ and $x(\xi_1)$ in the equation (2) are usually known. One wishes to determine the function u defined in $[\xi_0, \xi_1]$, which minimizes the cost function J:

$$J(u) = q(x(\xi_1)) \tag{3}$$

Here q is a given function from the same class as f. The cost function may represent displacement of a chosen point, volume of an element, etc.

These types of problems as presented shortly above can be solved using one of the variants of the Pontryagin maximum principle (Alekseev *et al.* 1987; Hartl *et al.* 1995; Ioffe and Tikhomirov 1979). The next section proves that such approach can be applied to shape optimization of many mechanical systems. See also the works of (Szefer and Mikulski 1978, 1984; Atanackovic 2001; Mitkowski and Skruch 2001; Skruch 2001). As an example we will consider a single span beam. Then we will show how to implement numerical solution of the problem. A numerical study of the existence of local minimum for the optimization problem will be given in Section 4. Other examples will be presented in Section 5 before summing up in Section 6.

2. Problem formulation

Let us consider a single span beam with rectangular cross–section working under selfweight (Fig. 1). The statics of the beam can be described using the following equation

$$\frac{d^2}{d\xi^2}\left[EI\,\frac{d^2 y(\xi)}{d\xi^2}\right] = -u(\xi) \tag{4}$$

where $\xi \in (0, l)$, $u(\xi) = \gamma b h(\xi)$, $y(\xi)$ represents vertical displacement of the beam along the interval $(0, l)$, l stands for the length of the beam, b is the width of the beam's cross–section, h is the height of the cross–section, E is the Young's module, I is the moment of inertia of the crosssection, γ is the volume mass density of the beam material.

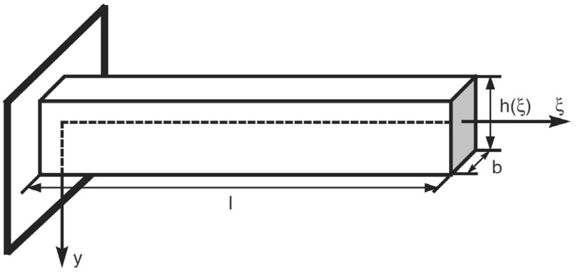

Fig. 1. Single span beam under self-weight

For state notation of the equation (4) we introduce the vector

$$x(\xi) = [x_1(\xi) x_2(\xi) x_3(\xi) x_4(\xi)]^T \tag{5}$$

where

$$\begin{cases} x_1(\xi) = y(\xi) \\ x_2(\xi) = y'(\xi) \\ x_3(\xi) = EIy''(\xi) = E\dfrac{bu(\xi)^3}{12} y''(\xi) \\ x_4(\xi) = EIy'''(\xi) \end{cases} \tag{6}$$

and additionally we define the vector

$$f(x(\xi), u(\xi)) = \begin{bmatrix} x_2(\xi) \\ \dfrac{12x_3(\xi)}{Ebu(\xi)^3} \\ x_4(\xi) \\ -u(\xi) \end{bmatrix} \tag{7}$$

Then our system can be written shortly in the classical form (1) with the boundary conditions

$$x_1(0) = x_2(0) = x_3(l) = x_4(l) = 0 \tag{8}$$

Here, the values $x_3(0)$, $x_4(0)$, $x_1(l)$ and $x_2(l)$ are unknown.

Side conditions concerning strength constraints and geometry are imposed on the dimensions of the cross–section, so that

$$U_{ad} = \{u : H_1 \leq u(\xi) \leq H_2, \xi \in [0, l]\} \tag{9}$$

The deflection at the end point of the beam is the optimality criterion

$$J(u) = x_1(l) \tag{10}$$

The cost function (10) can be also expressed in the following form

$$J(u) = x_1(l) = \int_0^l x_2(\xi)d\xi \tag{11}$$

Using the subsitution

$$x_0(\xi) = \int_0^\xi x_2(\tau)d\tau, \quad x_0(0) = 0 \tag{12}$$

we can add to our system (1), (5), (7), (8) the additional differential equation

$$\frac{dx_0}{d\xi} = x_2, \quad x_0(0) = 0 \tag{13}$$

We want to determine such $u \in U_{ad}$, which minimizes the functional (10) and satisfies the state equations (1), (5), (7), (12) with the boundary conditions (8).

In order to solve the formulated problem, we use the Pontryagin maximum principle. Therefore we introduce the Hamiltonian

$$H(\lambda_0, \lambda, x_0, x, u) = [\lambda_0 \ \lambda_1 \ \lambda_2 \ \lambda_3 \ \lambda_4] \begin{bmatrix} x_2 \\ x_2 \\ \dfrac{12x_3}{Ebu^3} \\ x_4 \\ -u \end{bmatrix} = \tag{14}$$

$$= \lambda_0 x_2 + \lambda_1 x_2 + \lambda_2 \frac{12x_3}{Ebu^3} + \lambda_3 x_4 + \lambda_4 u$$

where the variable $\lambda_0 \le 0$ and the vector of adjoint variables $\lambda = [\lambda_1, \lambda_2, \lambda_3, \lambda_4]$ satisfies the equation

$$\frac{d\lambda^T}{d\xi} = -\frac{\partial H}{\partial x} = \begin{bmatrix} 0 \\ -\lambda_0 - \lambda_1 \\ -\lambda_2 \dfrac{12x_3}{Ebu^3} \\ -\lambda_3 \end{bmatrix} \tag{15}$$

Transversality conditions lead to the following boundary values

$$\lambda_3(0) = 0,\ \lambda_4(0) = 0 \tag{16}$$

$$\lambda_1(l) = 0,\ \lambda_2(l) = 0 \tag{17}$$

Additionally, the following inequality must be satisfied

$$|\lambda_0| + |\lambda_1(0)| + |\lambda_2(0)| + |\lambda_3(l)| + |\lambda_4(l)| > 0 \tag{18}$$

According to the Pontryagin maximum principle for the optimal control u^* there is

$$H(\lambda_0^*, \lambda^*, x_0^*, x^*, u^*) = \max_{u \in U_{ad}} H(\lambda_0^*, \lambda^*, x_0^*, x^*, u^*) \tag{19}$$

The optimal control u^* can be obtained from the condition $\partial H / \partial u = 0$ with the help of the system of the adjoint equations (15) and the boundary conditions (16), (17).

We assume that $\lambda_0 = -1$. Then from (15) we can obtain

$$\lambda_1(\xi) = 0,\ \lambda_2(\xi) = \xi - l \tag{20}$$

Invoking the condition $\partial H / \partial u = 0$ we have the equation

$$u(\xi) = \sqrt[4]{(l-\xi)\frac{36 x_3(\xi)}{Eb\lambda_4(\xi)}} \tag{21}$$

from which we shall obtain the optimal control u^*. In the formulated optimization problem the constraints (9) cause that not the whole space is an admissible region. Because

$$\lim_{\xi \to 0_+} \sqrt[4]{(l-\xi)\frac{36 x_3(\xi)}{Eb\lambda_4(\xi)}} = +\infty \tag{22}$$

$$\lim_{\xi \to l_-} \sqrt[4]{(l-\xi)\frac{36 x_3(\xi)}{Eb\lambda_4(\xi)}} = 0 \tag{23}$$

then the optimal solution has the final form

$$u^*(\xi) = \begin{cases} H_2, & \text{for} \quad \xi \in [0, \xi_a] \\ \sqrt[4]{(l-\xi)\dfrac{36 x_3(\xi)}{Eb\lambda_4(\xi)}}, & \text{for} \quad \xi \in (\xi_a, \xi_b] \\ H_1, & \text{for} \quad \xi \in (\xi_b, l] \end{cases} \tag{24}$$

3. Problem solution

To find effectively the optimal control $u^*(\xi)$, it is necessary to solve the system which consists of nonlinear ordinary differential equations of the first order with the boundary conditions defined at initial and end points. The solution of this system is possible only in a numerical way.

The following algorithm has been implemented in MATLAB/Simulink™ environment. The algorithm for numerical solution of the shape optimization problem for the single span beam with rectangular cross–section uses simple shooting method.

Assumptions: The system (1), (5), (7), (8) has a solution and the optimal control exists.

Step 1: For arbitrarily chosen values $x_3(0)$ and $x_4(0)$ solve the problem in the interval $[0, \xi_a]$ using model created in Simulink™. In this model time works as geometrical variable ξ.
Then find ξ_a such that $u^*(\xi_a) = H_2$ using the equation (21).

Step 2: Based on the results from previous step determine the end conditions $x(\xi_a)$. They will be used as initial conditions in the next step.

Step 3: Find ξ_b such that $u^*(\xi_b) = H_1$ using the equation (21).

Step 4: Solve the problem in the interval $(\xi_a, \xi_b]$ using model created in Simulink™ and determine the end conditions $x(\xi_b)$. These values will be used as initial conditions in the next step.

Step 5: Solve the problem in the interval $(\xi_b, l]$ using model created in Simulink™.

Step 6: Calculate the norm $|x_3(l)| + |x_4(l)|$ and compare it with 0. First approximation for the optimal control $u^*(\xi)$ can be obtained from (24).

Step 7: Based on the norm $|x_3(l)| + |x_4(l)|$ recalculate the points $x_3(0)$ and $x_4(0)$ using the Nelder–Mead simplex (direct search) method (fmins function included in MATLAB™).

Step 8: The steps 1–8 are repeated many times until $|x_3(l)| + |x_4(l)| < \varepsilon$.

Simulation effects are shown in Figures 2–7. Figure 2 presents the optimal height $h(\xi)$ of the crosssection. Figure 3 presents the optimal shape of the beam. Figures 4 and 5 illustrate the state variables $x_1(\xi)$, $x_2(\xi)$, $x_3(\xi)$ and $x_4(\xi)$ along the interval $[0, l]$, Figures 6 and 7 show the set of adjoint variables λ_i, $i = 0, 1, 2, 3, 4$. Calculation were made for the following data: $l = 2.0$ m, $b = 0.1$ m, $H_1 = 0.1$ m, $H_2 = 0.2$ m, $E = 2.1 \cdot 10^{11}$ N/m^2, $\gamma = 76\,500$ N/m^3.

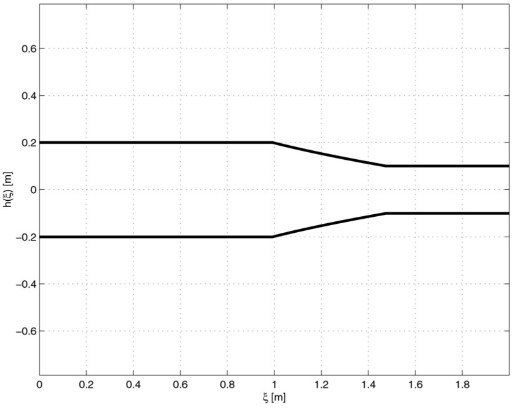

Fig. 2. The height of the cross–section, $\xi_a = 0.99$ m, $\xi_b = 1.47$ m

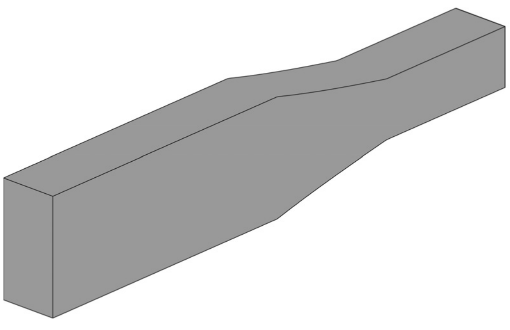

Fig. 3. Optimal shape of the beam

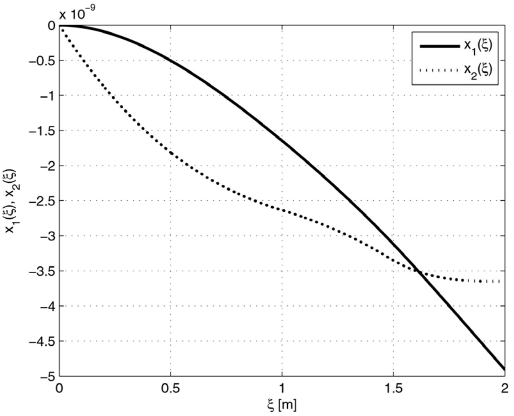

Fig. 4. The state variables $x_1(\xi)$, $x_2(\xi)$

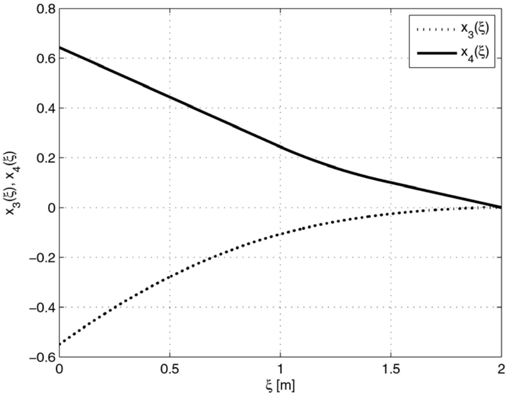

Fig. 5. The state variables $x_3(\xi)$, $x_4(\xi)$

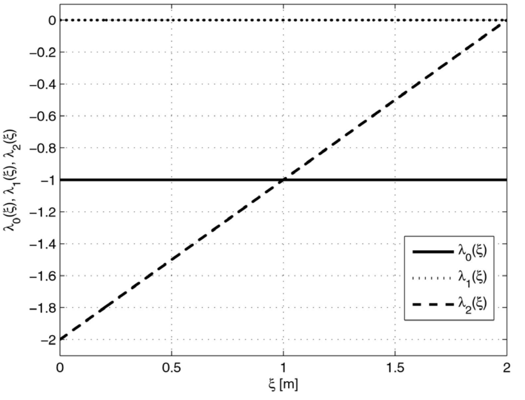

Fig. 6. The adjoint variables λ_0, λ_1, λ_2

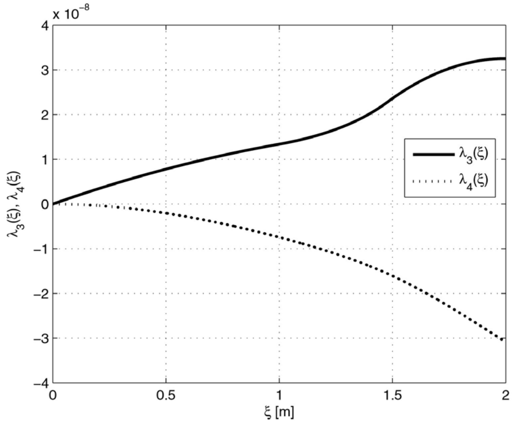

Fig. 7. The adjoint variables λ_3, λ_4

4. Study of the existence of local minimum

The Pontryagin maximum principle gives a necessary condition for an optimum. It does not assure that the solution of the problem really exists and is unique. A general proof of existence, uniqueness and stability is usually impossible.

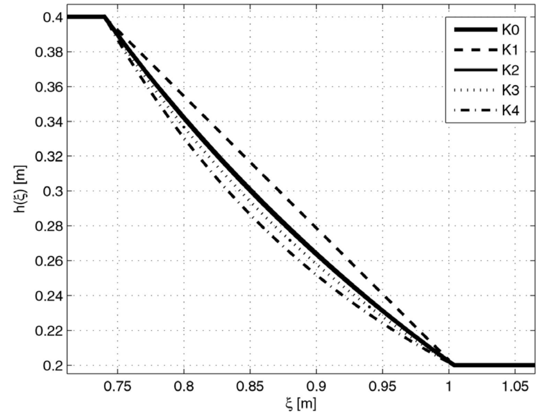

Fig. 8. The candidate for optimal shape K0 and the shapes in the neighborhood K1, K2, K3 and K4

This needs an extensive study and will not be provided in this paper. However, these issues are important from the theoretical point of view. Some attempts have been made by (Skruch 2001) how to handle this numerically. In some neighborhood of the candidate for optimal shape K0 (see Fig. 8) we choose other shapes K1, K2, K3 and K4. These shapes are described by the following equations:

$$\text{K1: } y = -0.7576\xi + 0.9606 \tag{25}$$

$$\text{K2: } y = 0.5628\xi^{-1} - 0.3606 \tag{26}$$

$$\text{K3: } y = 0.2398\xi^{-2} - 0.0379 \tag{27}$$

$$\text{K4: } y = 0.1352\xi^{-3} + 0.0664 \tag{28}$$

Then for every shape we need to calculate the cost function J that is the deflection at the end point of the beam. Figure 9 presents results of this calculation; for the shapes in the neighborhood of the candidate for optimal shape K0 we obtain worse values of the cost function J.

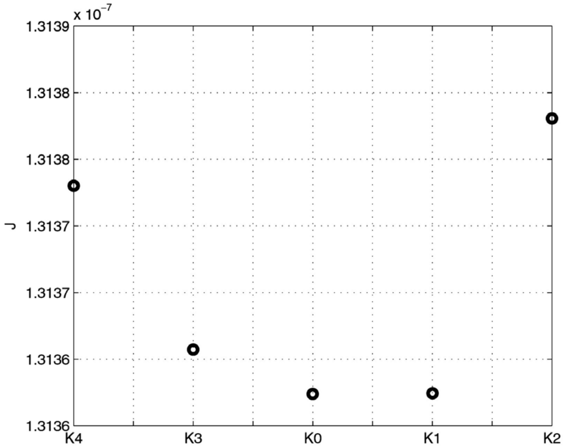

Fig. 9. The cost function J in the neighborhood of the candidate for optimal shape K0

Other neighborhood of the candidate for optimal shape is presented in Figure 10. The candidate for optimal shape is depicted using bold line.

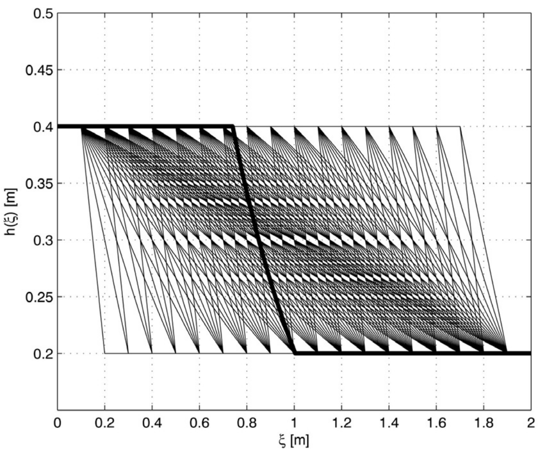

Fig. 10. The neighborhood of the candidate for optimal shape

The neighborhood contains shapes in the form of straight lines with different points ξ_a and ξ_b (thin lines). Then for every shape we calculated the deflection of the beam at the end point. The results of these calculations are shown in Figure 11.

For the shapes in the neighborhood we obtain worse values of the cost function J than for the shape K0.

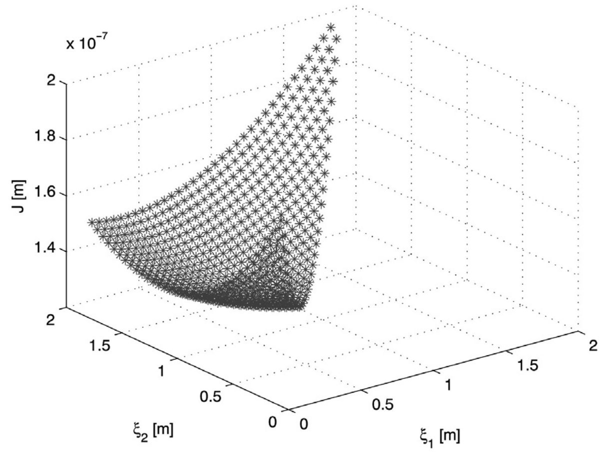

Fig. 11. The cost function J in the neighborhood of the candidate for optimal shape

5. Other examples

The methodology presented in this paper can be used for solving other types of shape optimization problems. For example, instead of rectan gular cross–section we can choose circular one.

I cross–section, etc., see (Szefer and Mikulski 1978; Skruch 2001). Also, the algorithm presented in Section 4 can be easily adapted to other types of problems. Strength optimization of elastic arches with I cross–section (Szefer and Mikulski 1984) and rectangular cross–section (Szefer and Mikulski 1985) can be solved using the presented approach. The Pontryagin maximum principle can be also used for determining the optimal shape of the lightest rotating rod, stable against buckling (Atanackovic 2001). The principle gives also good results for structural optimization problems (Szefer *et al.* 2005).

Figure 12 presents the single span beam with I cross–section. The statics of the beam can be described using the equation (1). The output of the presented algorithm is shown in Figure 13.

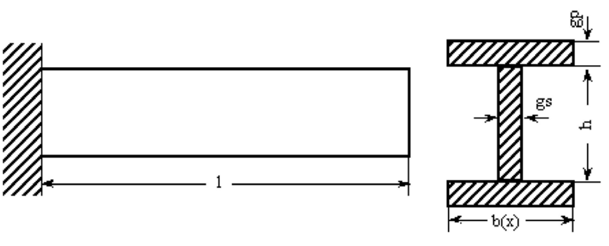

Fig. 12. Single span beam with I cross–section

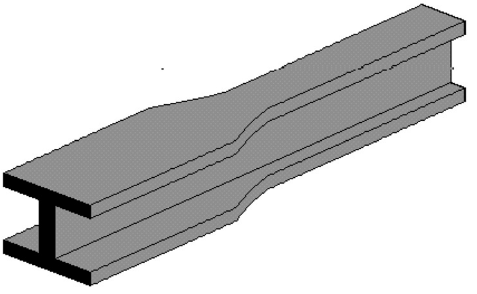

Fig. 13. Optimal shape of the beam with I cross–section

The next figures (Figs. 14–19) illustrate the results obtained for clamped beam with rectangular cross–section. Both the width and height of the cross–section can be taken as the control variable.

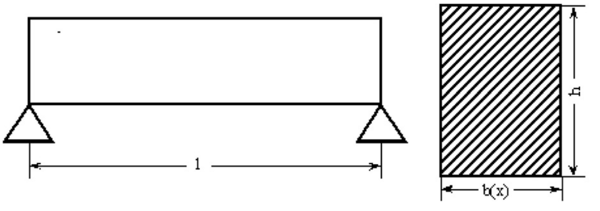

Fig. 14. Clamped beam with rectangular cross–section
(the width of the cross–section is taken as the control variable)

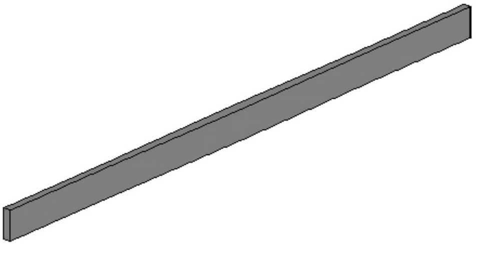

Fig. 15. Optimal shape of the clamped beam

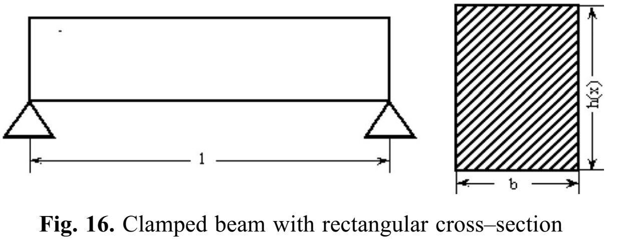

Fig. 16. Clamped beam with rectangular cross–section
(the height of the cross–section is taken as the control variable)

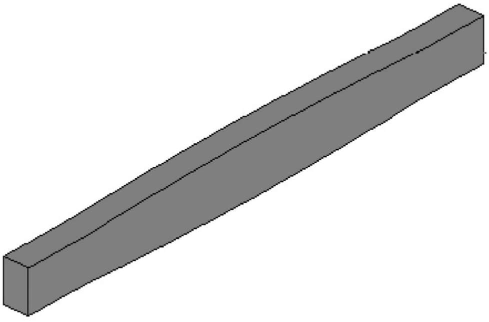

Fig. 17. Optimal shape of the clamped beam

6. Conclusions

We have investigated a shape optimization problem. As it has been shown in Section 3 the problem is not always trivial and the general proof can be very difficult. By using effective Pontryagin's method of optimization, the numerical algorithm has been designed and implemented (Section 4). The simulation results show the effectiveness of the proposed method.

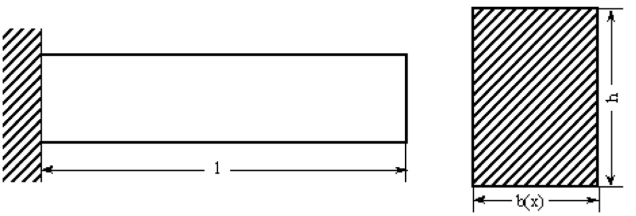

Fig. 18. Single span beam with rectangular cross–section
(the width of the cross–section is taken as the control variable)

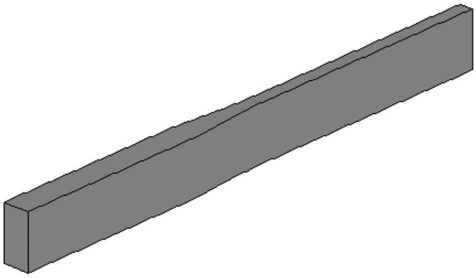

Fig. 19. Optimal shape of the beam

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