

MAGNETIC DIRICHLET LAPLACIAN IN CURVED WAVEGUIDES

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Abstract. For a two-dimensional curved waveguide, it is well known that the spectrum of the Dirichlet Laplacian is unstable with respect to waveguide deformations. This means that if the waveguide is a straight strip then the spectrum of the Dirichlet Laplacian is purely essential. From the other hand, the perturbation of the straight strip produces eigenvalues below the essential spectrum. In this paper, the Dirichlet–Laplace operator with a magnetic field is considered. We explicitly prove that the spectrum of the magnetic Laplacian is stable under small but non-local deformations of the waveguide.

Keywords: magnetic Schrödinger operators, essential spectrum, discrete spectrum.

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1. INTRODUCTION

It has long been known that an appropriate bending of a two dimensional quantum waveguide induces the existence of bound states, [8, 9] and [5]. From a mathematical point of view, this means that the Dirichlet Laplacian on a smooth asymptotically straight planar waveguide has at least one isolated eigenvalue below the threshold of the essential spectrum. Similar results have been obtained for a locally deformed waveguide, which corresponds to adding a small “bump” to the straight waveguide, see [4] and [2]. As a result at least one isolated eigenvalue appears below the essential spectrum for any nonzero curvature, satisfying certain regularity properties.

As is well-known, that a magnetic field, even a local one, can significantly affect the behaviour of waveguide systems, in particular the existence of a geometrically induced discrete spectrum. While a particle confined in a fixed-profile tube with a hard-wall boundary can exist in localized states whenever the tube is bent or locally deformed (and asymptotically straight), cf. [7] for a comprehensive review of quantum waveguide theory, the presence of a local magnetic field can destroy such a discrete spectrum.

The analogous effect of bound state existence resulting from the geometry of the interaction support has been observed in a class of singular Schrödinger operators, usually dubbed leaky quantum wires, with attractive contact interaction supported by a curve [7]. It was established in [1] that the presence of a local magnetic field can again destroy such a discrete spectrum. Another similar result has been established in [3], where the authors consider the magnetic Schrödinger operator with a non-negative potential supported over a strip which is a local deformation of a straight one, and the magnetic field is assumed to be nonzero and local. The object of our interest in this paper will be the magnetic Dirichlet Laplacian, which is constructed as follows. Consider the closed quadratic form

$$\|(i\nabla + A)u\|_{L^2(\Omega)}, \quad u \in \mathcal{H}_0^1(\Omega),$$

where the real-value function A is a vector potential.

By the first representation theorem [10] there is a unique nonnegative self-adjoint operator

$$H_\Omega(A) = (i\nabla + A)^2 \tag{1.1}$$

associated with this form. In the case studied in this work the boundary of Ω is smooth, and the magnetic field is compactly supported which allows us to define the domain of the operator (1.1) as $\mathcal{H}_0^1(\Omega) \cap \mathcal{H}^2(\Omega)$.

Our main results are presented in Section 3. Here we prove the stability of the essential spectrum of the operator (1.1) with respect to the situation when the strip is a smooth asymptotically straight planar waveguide and the magnetic field is non-zero and vanishing at infinity. We also give a sufficient condition for the absence of the discrete spectrum. Let us mention that the similar results have been obtained in [6], but the authors in that work deal only with compactly supported perturbations of the straight tube. In the present work it has been shown that the vanishing and non-zero magnetic field can destroy the discrete spectrum even if the perturbation of the tube is not local.

2. CURVED WAVEGUIDES

The domain Ω studied in our paper is assumed to be a curved planar strip of a width d . Its points are described by the curvilinear coordinates $s \in \mathbb{R}$, $u \in (0, d)$ as follows:

$$\begin{aligned} x &= x(s, u) = a(s) - u\dot{b}(s), \\ y &= y(s, u) = b(s) + u\dot{a}(s), \end{aligned} \tag{2.1}$$

where dot marks the derivative with respect to s and a, b are sufficiently smooth functions characterizing the reference curve $\Gamma = \{(a(s), b(s)) : s \in \mathbb{R}\}$. We assume

$$\dot{a}(s)^2 + \dot{b}(s)^2 = 1, \tag{2.2}$$

so s is the arc length of Γ , and $u \in (0, d)$ is the distance of the point (x, y) from Γ . It is useful to introduce the signed curvature $\gamma(s)$ of Γ ,

$$\gamma(s) = \dot{b}(s)\ddot{a}(s) - \dot{a}(s)\ddot{b}(s). \tag{2.3}$$

The characterization of the region (2.1) by the curvilinear coordinate only makes sense if the latter can be uniquely defined, which imposes two different restrictions. First, the transverse size must not be too large, the inequality $d|\gamma(s)| < 1$ must hold at every point of the curve. In addition, the region Ω does not intersect itself. For obvious reasons, Ω is called a curved quantum waveguide.

Let us create the following identity that we will use later. Given (2.3) and (2.2) we have

$$\begin{aligned}\gamma(s)^2 &= (\dot{b}(s)\ddot{a}(s) - \dot{a}(s)\ddot{b}(s))^2 \\ &= \ddot{a}(s)^2 - \ddot{a}(s)^2\dot{a}(s)^2 + \ddot{b}(s)^2\dot{a}(s)^2 - 2\ddot{a}(s)\dot{b}(s)\ddot{b}(s)\dot{a}(s).\end{aligned}\quad (2.4)$$

With respect to identity (2.2), we have

$$\dot{a}(s)\ddot{a}(s) + \dot{b}(s)\ddot{b}(s) = 0. \quad (2.5)$$

Therefore, (2.4) implies

$$\gamma(s)^2 = \ddot{a}(s)^2 - \ddot{b}(s)^2\dot{b}(s)^2 + \ddot{b}(s)^2\dot{a}(s)^2 + 2\ddot{b}(s)^2\dot{b}(s)^2 = \ddot{a}(s)^2 + \ddot{b}(s)^2. \quad (2.6)$$

During our work will need the following conditions:

(a)

$$|\gamma(s)|, |\dot{\gamma}(s)|, |\ddot{\gamma}(s)|, |\ddot{\gamma}(s)| \leq \frac{\beta}{1+s^2}, \quad s \in \mathbb{R},$$

where $\beta > 0$ is some constant to be described later.

(b) Suppose that the magnetic vector potential $A(x, y) = (a_1(x, y), a_2(x, y))$ is such that for $j = 1, 2$ the functions $a_j, \frac{\partial a_j}{\partial x}, \frac{\partial a_j}{\partial y}$ vanish for large values of x .

3. MAIN RESULTS

We shall establish the stability of the essential spectrum and the absence of the discrete spectrum below the threshold of the essential spectrum of the operator (1.1) defined on an asymptotically straight curved waveguide under the assumptions (a) and (b). Let us start with the absence of the discrete spectrum due to a non-zero magnetic field. Before stating the result, it is useful to mention that later we shall show that the essential spectrum of the operator (1.1) coincides with the half-line $[\frac{\pi^2}{d^2}, \infty)$.

3.1. ABSENCE OF THE DISCRETE SPECTRUM

Theorem 3.1. *Let $B = \text{rot}(A) \in C^1(\mathbb{R}^2)$ be a real-valued magnetic field which is non-trivial in Ω . Assume that assumptions (a) and (b) hold. There exists a positive number β_0 such that for $\beta \in (0, \beta_0)$ the discrete spectrum of the operator (1.1) below $\frac{\pi^2}{d^2}$ is empty.*

Proof. Let us denote by q the quadratic form of the operator (1.1). Then

$$q_{\Omega}^A(\varphi) = \int_{\Omega} \left(\left| i \frac{\partial \varphi}{\partial x} + a_1 \varphi \right|^2 + \left| i \frac{\partial \varphi}{\partial y} + a_2 \varphi \right|^2 \right) dx dy, \quad (3.1)$$

where $A = (a_1, a_2)$.

We shall restrict ourselves from the beginning by the requirement

$$\beta_0 < \frac{1}{d}$$

which guarantees that

$$1 + u\gamma(s) \geq 1 - d\beta_0 > 0.$$

Define the unitary operator

$$U : L^2(\Omega) \rightarrow L^2(\Omega_0), \quad \Omega_0 = \mathbb{R} \times (0, d), \quad (3.2)$$

which for any $\varphi \in L^2(\Omega)$ acts as

$$\psi(s, u) := (U\varphi)(s, u) = \sqrt{1 + u\gamma(s)} \varphi(a(s) - u\dot{b}(s), b(s) + u\dot{a}(s)). \quad (3.3)$$

Moreover, if $\varphi|_{\partial\Omega} = 0$, then the same is true for ψ : $\psi|_{\partial\Omega_0} = 0$.

Let us now follow the calculations of [6]. We can check that the Jacobian

$$\frac{\partial(x, y)}{\partial(s, u)} = 1 + u\gamma(s) \quad (3.4)$$

and

$$\begin{aligned} \frac{\partial \varphi}{\partial x} &= (1 + u\gamma)^{-1} \left(\dot{a} \frac{\partial}{\partial s} - (\dot{b} + u\ddot{a}) \frac{\partial}{\partial u} \right) \left(\frac{\psi}{\sqrt{1 + u\gamma}} \right), \\ \frac{\partial \varphi}{\partial y} &= (1 + u\gamma)^{-1} \left(\dot{b} \frac{\partial}{\partial s} - (\dot{a} - u\ddot{b}) \right) \left(\frac{\psi}{\sqrt{1 + u\gamma}} \right). \end{aligned} \quad (3.5)$$

Then, using the notation

$$\begin{aligned} \tilde{A}(s, u) &= (\tilde{a}_1(s, u), \tilde{a}_2(s, u)) \\ &= (a_1(a(s) - u\dot{b}(s), b(s) + u\dot{a}(s)), a_2(a(s) - u\dot{b}(s), b(s) + u\dot{a}(s))) \end{aligned} \quad (3.6)$$

we rewrite the quadratic form $q_\Omega^A(\varphi)$, $\varphi \in \mathcal{H}_0^1(\Omega)$, as follows:

$$\begin{aligned} q_\Omega^A(\varphi) = \int_{\Omega_0} & \left(\left| \left(\frac{i}{1+u\gamma} \left(\dot{a} \frac{\partial}{\partial s} - (\dot{b} + u\ddot{a}) \frac{\partial}{\partial u} \right) + \tilde{a}_1 \right) \left(\frac{\psi}{\sqrt{1+u\gamma}} \right) \right|^2 \right. \\ & + \left| \left(\frac{i}{1+u\gamma} \left(\dot{b} \frac{\partial}{\partial s} + (\dot{a} - u\ddot{b}) \frac{\partial}{\partial u} \right) + \tilde{a}_2 \right) \left(\frac{\psi}{\sqrt{1+u\gamma}} \right) \right|^2 \Bigg) \\ & \cdot (1+u\gamma) ds du. \end{aligned} \quad (3.7)$$

Using (2.5) and (2.6) expression (3.7) performs

$$\begin{aligned} q_\Omega^A(\varphi) = \int_{\Omega_0} & \left(\frac{1}{(1+u\gamma)^2} \left| \frac{\partial \psi}{\partial s} \right|^2 + \frac{i(\dot{a}\tilde{a}_1 + \dot{b}\tilde{a}_2)}{1+u\gamma} \left(\frac{\partial \psi}{\partial s} \bar{\psi} - \psi \frac{\partial \bar{\psi}}{\partial s} \right) + \left| \frac{\partial \psi}{\partial u} \right|^2 \right. \\ & + \frac{i(-(\dot{b} + u\ddot{a})\tilde{a}_1 + (\dot{a} - u\ddot{b})\tilde{a}_2)}{1+u\gamma} \left(\frac{\partial \psi}{\partial u} \bar{\psi} - \psi \frac{\partial \bar{\psi}}{\partial u} \right) \\ & - \frac{u\dot{\gamma}}{2(1+u\gamma)^3} \left(\psi \frac{\partial \bar{\psi}}{\partial s} + \frac{\partial \psi}{\partial s} \bar{\psi} \right) - \frac{\gamma}{2(1+u\gamma)} \left(\psi \frac{\partial \bar{\psi}}{\partial u} + \frac{\partial \psi}{\partial u} \bar{\psi} \right) \\ & \left. + \left(\frac{u^2(\dot{\gamma})^2}{4(1+u\gamma)^4} + \frac{\gamma^2}{4(1+u\gamma)^2} + \tilde{a}_1^2 + \tilde{a}_2^2 \right) |\psi|^2 \right) ds du. \end{aligned} \quad (3.8)$$

We write the right-hand side of (3.8) as a perturbation of the form $\tilde{q}_\Omega^A$ as follows:

$$q_\Omega^A(\varphi) = \tilde{q}_\Omega^A - I(\psi), \quad (3.9)$$

where

$$\begin{aligned} \tilde{q}_\Omega^A(\psi) &= \int_{\Omega_0} \left| i \frac{\partial \psi}{\partial s} + (\dot{a}\tilde{a}_1 + \dot{b}\tilde{a}_2)\psi \right|^2 + \left| i \frac{\partial \psi}{\partial u} + (-\dot{b}\tilde{a}_1 + \dot{a}\tilde{a}_2)\psi \right|^2 ds du, \\ I(\psi) &= \int_{\Omega_0} \left(\frac{2u\gamma + u^2\gamma^2}{1+u\gamma} \left| \frac{\partial \psi}{\partial s} \right|^2 + iu\gamma(\dot{a}\tilde{a}_1 + \dot{b}\tilde{a}_2) \left(\frac{\partial \psi}{\partial s} \bar{\psi} - \psi \frac{\partial \bar{\psi}}{\partial s} \right) \right. \\ &+ iu \left(-\gamma\dot{b}\tilde{a}_1 + \gamma\dot{a}\tilde{a}_2 + \frac{\ddot{a}\tilde{a}_1 - \ddot{b}\tilde{a}_2}{1+u\gamma} \right) \left(\frac{\partial \psi}{\partial u} \bar{\psi} - \psi \frac{\partial \bar{\psi}}{\partial u} \right) \\ &+ \frac{u\dot{\gamma}}{2(1+u\gamma)^3} \left(\psi \frac{\partial \bar{\psi}}{\partial s} + \frac{\partial \psi}{\partial s} \bar{\psi} \right) + \frac{\gamma}{2(1+u\gamma)} \left(\psi \frac{\partial \bar{\psi}}{\partial u} + \frac{\partial \psi}{\partial u} \bar{\psi} \right) \\ &\left. - \left(\frac{u^2(\dot{\gamma})^2}{4(1+u\gamma)^4} + \frac{\gamma^2}{4(1+u\gamma)^2} \right) |\psi|^2 \right) ds du, \end{aligned}$$

Now let us estimate $I(\psi)$. We will use the notation $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\mathbb{R})}$. Since $u \in (0, d)$, then given (2.2) and (2.6) we get

$$\begin{aligned}
& |I(\psi)| \\
& \leq \frac{2d + d^2\|\gamma\|_\infty}{1 - d\|\gamma\|_\infty} \int_{\Omega_0} |\gamma| \left| \frac{\partial\psi}{\partial s} \right|^2 ds du + 2d\sqrt{\|\tilde{a}_1\|_\infty^2 + \|\tilde{a}_2\|_\infty^2} \int_{\Omega_0} |\gamma||\psi| \left| \frac{\partial\psi}{\partial s} \right| ds du \\
& \quad + 2d\sqrt{\|\tilde{a}_1\|_\infty^2 + \|\tilde{a}_2\|_\infty^2} \int_{\Omega_0} |\gamma||\psi| \left| \frac{\partial\psi}{\partial u} \right| ds du \\
& \quad + \frac{2d\sqrt{\|\tilde{a}_1\|_\infty^2 + \|\tilde{a}_2\|_\infty^2}}{1 - d\|\gamma\|_\infty} \int_{\Omega_0} |\gamma||\psi| \left| \frac{\partial\psi}{\partial u} \right| ds du \\
& \quad + \frac{d}{(1 - d\|\gamma\|_\infty)^3} \int_{\Omega_0} |\dot{\gamma}||\psi| \left| \frac{\partial\psi}{\partial s} \right| ds du + \frac{1}{1 - d\|\gamma\|_\infty} \int_{\Omega_0} |\gamma||\psi| \left| \frac{\partial\psi}{\partial u} \right| ds du \\
& \quad + \frac{d^2}{4(1 - d\|\gamma\|_\infty)^4} \int_{\Omega_0} |\dot{\gamma}|^2 |\psi|^2 ds du + \frac{1}{4(1 - d\|\gamma\|_\infty)^2} \int_{\Omega_0} \gamma^2 |\psi|^2 ds du \tag{3.10} \\
& \leq \left(\frac{d(2 + d\|\gamma\|_\infty)}{1 - d\|\gamma\|_\infty} + d\sqrt{\|\tilde{a}_1\|_\infty^2 + \|\tilde{a}_2\|_\infty^2} + \frac{d}{2(1 - d\|\gamma\|_\infty)^3} \right) \int_{\Omega_0} (|\gamma| + |\dot{\gamma}|) \left| \frac{\partial\psi}{\partial s} \right|^2 ds du \\
& \quad + \left(d\sqrt{\|\tilde{a}_1\|_\infty^2 + \|\tilde{a}_2\|_\infty^2} + \frac{2d\sqrt{\|\tilde{a}_1\|_\infty^2 + \|\tilde{a}_2\|_\infty^2} + 1}{2(1 - d\|\gamma\|_\infty)} \right) \int_{\Omega_0} |\gamma| \left| \frac{\partial\psi}{\partial u} \right|^2 ds du \\
& \quad + \left(2d\sqrt{\|\tilde{a}_1\|_\infty^2 + \|\tilde{a}_2\|_\infty^2} + \frac{2d\sqrt{\|\tilde{a}_1\|_\infty^2 + \|\tilde{a}_2\|_\infty^2} + 1}{2(1 - d\|\gamma\|_\infty)} + \frac{d}{2(1 - d\|\gamma\|_\infty)^3} \right. \\
& \quad \left. + \frac{d^2}{4(1 - d\|\gamma\|_\infty)^4} + \frac{1}{4(1 - d\|\gamma\|_\infty)^2} \right) \int_{\Omega_0} (|\gamma| + \gamma^2 + |\dot{\gamma}| + \dot{\gamma}^2) |\psi|^2 ds du.
\end{aligned}$$

Let us denote

$$\alpha_1 := \max \left\{ \frac{d(2 + d\|\gamma\|_\infty)}{1 - d\|\gamma\|_\infty} + d\sqrt{\|\tilde{a}_1\|_\infty^2 + \|\tilde{a}_2\|_\infty^2} + \frac{d}{2(1 - d\|\gamma\|_\infty)^3}, \right. \tag{3.11}$$

$$\left. d\sqrt{\|\tilde{a}_1\|_\infty^2 + \|\tilde{a}_2\|_\infty^2} + \frac{2d\sqrt{\|\tilde{a}_1\|_\infty^2 + \|\tilde{a}_2\|_\infty^2} + 1}{2(1 - d\|\gamma\|_\infty)} \right\},$$

$$\begin{aligned}
\alpha_2 := & 2d\sqrt{\|\tilde{a}_1\|_\infty^2 + \|\tilde{a}_2\|_\infty^2} + \frac{2d\sqrt{\|\tilde{a}_1\|_\infty^2 + \|\tilde{a}_2\|_\infty^2} + 1}{2(1 - d\|\gamma\|_\infty)} + \frac{d}{2(1 - d\|\gamma\|_\infty)^3} \\
& + \frac{d^2}{4(1 - d\|\gamma\|_\infty)^4} + \frac{1}{4(1 - d\|\gamma\|_\infty)^2}. \tag{3.12}
\end{aligned}$$

Hence, the above inequality implies

$$|I(\psi)| \leq \alpha_1 \int_{\Omega_0} (|\gamma| + |\dot{\gamma}|) |\nabla \psi|^2 ds du + \alpha_2 \int_{\Omega_0} (|\gamma| + \gamma^2 + |\dot{\gamma}| + \dot{\gamma}^2) |\psi|^2 ds du.$$

In view of (3.9) and the above estimate, we arrive

$$q_{\Omega}^A(\varphi) \geq \tilde{q}_{\Omega}^A - \alpha_1 \int_{\Omega_0} (|\gamma| + |\dot{\gamma}|) |\nabla \psi|^2 ds du - \alpha_2 \int_{\Omega_0} (|\gamma| + \gamma^2 + |\dot{\gamma}| + \dot{\gamma}^2) |\psi|^2 ds du. \quad (3.13)$$

One can easily check that for any magnetic potential $\hat{A} = (\hat{a}_1, \hat{a}_2)$ the following pointwise inequality holds

$$|\nabla \psi|^2 \leq 2 \left| i \nabla \psi + \hat{A} \psi \right|^2 + 2 (\hat{a}_1^2 + \hat{a}_2^2) |\psi|^2, \quad \psi \in \mathcal{H}^1(\Omega_0).$$

In view of this fact and (3.13) we get

$$q_{\Omega}^A(\varphi) \geq \int_{\Omega_0} (1 - \rho_1^{\gamma}) |i \nabla \psi + \tilde{A} \psi|^2 ds du - \int_{\Omega_0} \rho_2^{\gamma} |\psi|^2 ds du, \quad (3.14)$$

where

$$\begin{aligned} \rho_1^{\gamma} &= 2\alpha_1 (|\gamma| + |\dot{\gamma}|), \\ \rho_2^{\gamma} &= 2\alpha_1 (\tilde{a}_1^2 + \tilde{a}_2^2) (|\gamma| + |\dot{\gamma}|) + \alpha_2 (|\gamma| + \gamma^2 + |\dot{\gamma}| + \dot{\gamma}^2). \end{aligned} \quad (3.15)$$

For the further proof we need the following Hardy-type inequality.

Lemma 3.2. *For any function $g \in \mathcal{H}_0^1(\Omega_0)$, the following estimate holds:*

$$\int_{\Omega_0} \left(f^2 |i \nabla g + \tilde{A} g|^2 - \frac{\pi^2}{d^2} f^2 |g|^2 \right) ds du \geq C_{\tilde{A}} \int_{\Omega_0} \frac{f^2}{1 + s^2} |g|^2 ds du + \int_{\Omega_0} f'' |g|^2 ds du,$$

where $C_{\tilde{A}} > 0$ is a constant, $\tilde{A} = (\tilde{a}_1, \tilde{a}_2)$ is the magnetic potential, $f = f(s) : \mathbb{R} \rightarrow (0, \infty)$ is a smooth function.

Let us return to the proof of the lower bound (3.14). Using Lemma 3.2 with $f(s) = (1 - \rho_1^{\gamma}(s))^{1/2}$ we get

$$\begin{aligned} q_{\Omega}^A(\varphi) &- \frac{\pi^2}{d^2} \int_{\Omega_0} (1 - \rho_1^{\gamma}(s)) |\psi|^2 ds du \\ &\geq C_{\tilde{A}} \int_{\Omega_0} \frac{(1 - \rho_1^{\gamma})}{1 + s^2} |\psi|^2 ds du - \frac{1}{4} \int_{\Omega_0} \frac{(2(1 - \rho_1^{\gamma}) \ddot{\rho}_1^{\gamma} + (\dot{\rho}_1^{\gamma})^2)}{(1 - \rho_1^{\gamma})^{3/2}} |\psi|^2 ds du \\ &\quad - \int_{\Omega_0} \rho_2^{\gamma} |\psi|^2 ds du. \end{aligned} \quad (3.16)$$

Let us choose a positive number $\beta_0 < \frac{1}{d}$ in assumption (a) such that for any $\beta \in (0, \beta_0)$ we have

$$C_{\tilde{A}} \frac{\rho_1^\gamma}{1+s^2} + \frac{1}{4} \frac{\left(2(1-\rho_1^\gamma)\ddot{\rho}_1^\gamma + (\dot{\rho}_1^\gamma)^2\right)}{(1-\rho_1^\gamma)^{3/2}} + \frac{\pi^2 \rho_1^\gamma}{d^2} + \rho_2^\gamma \leq \frac{C_{\tilde{A}}}{1+s^2}, \quad s \in \mathbb{R}.$$

This establishes that

$$q_\Omega^A(\varphi) - \frac{\pi^2}{d^2} \int_{\Omega_0} |\psi|^2 ds du = q_\Omega^A(\varphi) - \frac{\pi^2}{d^2} \int_{\Omega_0} |\varphi|^2 ds du \geq 0,$$

which proves that the discrete spectrum below $\frac{\pi^2}{d^2}$ of the operator (1.1) is empty. $\square$

3.2. STABILITY OF THE ESSENTIAL SPECTRUM

Theorem 3.3. *Suppose that the magnetic vector potential $A = (a_1, a_2)$ is such that for $j = 1, 2$ the functions a_j , $\frac{\partial a_j}{\partial x}$, $\frac{\partial a_j}{\partial y}$ are vanishing for large values of x . Then the essential spectrum of the operator (1.1) coincides with the half-line $[\frac{\pi^2}{d^2}, \infty)$.*

Proof. To prove that any non-negative number $\mu \geq \frac{\pi^2}{d^2}$ belongs to the essential spectrum of $H_\Omega(A)$, we will use Weyl's criterion [11, Thm. VII.12]: we have to find a sequence $\{\varphi_n\}_{n=1}^\infty \subset D(H_\Omega(A))$ of unit vectors, $\|\varphi_n\| = 1$, which converges weakly to zero and

$$\|H_\Omega(A)\varphi_n - \mu\varphi_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

holds.

First, let us rewrite operator (1.1) in the following form:

$$H_\Omega(A) = -\Delta + 2i \left(a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} \right) + i \left(\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} \right) + (a_1^2 + a_2^2).$$

Then

$$\begin{aligned} & \int_{\Omega} |H_\Omega(A)\varphi_n - \mu\varphi_n|^2 dx dy \\ &= \int_{\Omega} \left| -\Delta\varphi_n + 2i \left(a_1 \frac{\partial\varphi_n}{\partial x} + a_2 \frac{\partial\varphi_n}{\partial y} \right) + i \left(\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} \right) \varphi_n + (a_1^2 + a_2^2)\varphi_n - \mu\varphi_n \right|^2 dx dy \end{aligned}$$

and

$$\begin{aligned} & \int_{\Omega} |H_\Omega(A)\varphi_n - \mu\varphi_n|^2 dx dy \\ & \leq 4 \int_{\Omega} |-\Delta\varphi_n - \mu\varphi_n|^2 dx dy + 16 \int_{\Omega} \left| a_1 \frac{\partial\varphi_n}{\partial x} + a_2 \frac{\partial\varphi_n}{\partial y} \right|^2 dx dy \\ & \quad + 4 \int_{\Omega} \left| \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} \right|^2 |\varphi_n|^2 dx dy + 4 \int_{\Omega} (a_1^2 + a_2^2)^2 |\varphi_n|^2 dx dy. \end{aligned} \tag{3.17}$$

For each $\mu = \frac{\pi^2}{d^2} + k^2$, $k \in \mathbb{Z}$, we will separately estimate each integral in (3.17). Let us mention that, using the transformation (3.2), it was established in [8] that the Dirichlet Laplacian $-\Delta$ on Ω can be rewritten as the operator H_0 on $L^2(\Omega_0)$, which acts as

$$(H_0\psi)(s, u) = -\frac{\partial}{\partial s} \left(\frac{1}{(1+u\gamma(s))^2} \frac{\partial \psi}{\partial s} \right) - \frac{\partial^2 \psi}{\partial u^2} + W(s, u)\psi,$$

$$W(s, u) := -\frac{\gamma^2(s)}{4(1+u\gamma(s))^2} + \frac{u\ddot{\gamma}(s)}{2(1+u\gamma(s))^3} - \frac{5}{4} \frac{u^2\dot{\gamma}^2(s)}{(1+u\gamma(s))^4}$$

and Dirichlet boundary conditions are imposed at $u = 0, d$.

Given the unitary equivalence, it is sufficient to estimate the first integral in (3.17) for the operator H_0 . We will construct the Weyl sequence $\varphi_n \in C_0^\infty(\Omega)$ in such a way that the corresponding mapping (3.2) sequence $\psi_n \in C_0^\infty(\Omega_0)$ is the following normalized sequence

$$\psi_n(s, u) = \sqrt{\frac{2}{dn}} f\left(\frac{s}{n}\right) e^{iks} \sin\left(\frac{\pi u}{d}\right), \quad n \in \mathbb{N}, \quad (3.18)$$

where $f \in C_0^\infty(\mathbb{R})$ is a smooth function with support in the interval $(1, 2)$ with L^2 norm equal to one.

We have

$$\begin{aligned} & \int_{\Omega_0} |H_0\psi_n - \mu\psi_n|^2 dx dy \\ &= \int_{\Omega_0} \left| -\frac{\partial}{\partial s} \left(\frac{1}{(1+u\gamma(s))^2} \frac{\partial \psi_n}{\partial s} \right) - \frac{\partial^2 \psi_n}{\partial u^2} + W(s, u)\psi_n - \mu\psi_n \right|^2 ds du \\ &= \frac{2}{dn} \int_{\Omega_0} \left| \frac{2u\dot{\gamma}}{(1+u\gamma)^3} \left(\frac{1}{n} f\left(\frac{s}{n}\right) + ikf\left(\frac{s}{n}\right) \right) \right. \\ &\quad \left. - \frac{1}{(1+u\gamma)^2} \left(\frac{1}{n^2} \ddot{f}\left(\frac{s}{n}\right) + \frac{2ik}{n} \dot{f}\left(\frac{s}{n}\right) - k^2 f\left(\frac{s}{n}\right) \right) \right. \\ &\quad \left. + \frac{\pi^2}{d^2} f\left(\frac{s}{n}\right) + W(s, u)f\left(\frac{s}{n}\right) - \mu f\left(\frac{s}{n}\right) \right|^2 \sin^2\left(\frac{\pi u}{d}\right) ds du \\ &= \frac{2}{dn} \int_{\Omega_0} \left| \frac{2u\dot{\gamma}}{(1+u\gamma)^3} \left(\frac{1}{n} f\left(\frac{s}{n}\right) + ikf\left(\frac{s}{n}\right) \right) - \frac{1}{(1+u\gamma)^2} \left(\frac{1}{n^2} \ddot{f}\left(\frac{s}{n}\right) + \frac{2ik}{n} \dot{f}\left(\frac{s}{n}\right) \right) \right. \\ &\quad \left. - \frac{k^2(2u\gamma + u^2\gamma^2)}{(1+u\gamma)^2} f\left(\frac{s}{n}\right) + W(s, u)f\left(\frac{s}{n}\right) \right|^2 \sin^2\left(\frac{\pi u}{d}\right) ds du. \end{aligned}$$

Because of assumption (a), it is easy to check that the right-hand side of the above estimate is $\mathcal{O}\left(\frac{1}{n^2}\right)$,

$$\int_{\Omega_0} |H_0\psi_n - \mu\psi_n|^2 dx dy = \mathcal{O}\left(\frac{1}{n^2}\right). \quad (3.19)$$

Let us pass to the second term in the right-hand side of (3.17) and estimate as the first $\int_{\Omega} a_1^2 \left| \frac{\partial \varphi_n}{\partial x} \right|^2 dx dy$. We pass to the curvilinear coordinates and use (3.4), (3.5) and the notations from (3.6). With the notation

$$\omega_n = \{ (x(s, u), y(s, u))_{s \in (n, 2n), u \in (0, d)} \}$$

one gets

$$\begin{aligned} & \int_{\Omega} a_1^2 \left| \frac{\partial \varphi_n}{\partial x} \right|^2 dx dy \\ &= \int_{\Omega_0} \tilde{a}_1^2 \left| (1 + u\gamma)^{-1} \left(\dot{a} \frac{\partial}{\partial s} - (\dot{b} + u\ddot{a}) \frac{\partial}{\partial u} \right) \left(\frac{\psi_n}{\sqrt{1 + u\gamma}} \right) \right|^2 (1 + u\gamma) ds du \\ &\leq \frac{\|\tilde{a}_1\|_{L^\infty(\omega_n)}^2}{(1 - d\|\gamma\|_\infty)^2} \int_{\Omega_0} \left| \frac{\dot{a}}{\sqrt{1 + u\gamma}} \frac{\partial \psi_n}{\partial s} - \frac{u\dot{a}\dot{\gamma}}{2(1 + u\gamma)^{3/2}} \psi_n \right. \\ &\quad \left. - \frac{(\dot{b} + u\ddot{a})}{\sqrt{1 + u\gamma}} \frac{\partial \psi_n}{\partial u} + \frac{(\dot{b} + u\ddot{a})\gamma}{2(1 + u\gamma)^{3/2}} \psi_n \right|^2 (1 + u\gamma) ds du \\ &\leq \frac{4\|\tilde{a}_1\|_{L^\infty(\omega_n)}^2}{(1 - d\|\gamma\|_\infty)^2} \int_{\Omega_0} \left(\frac{\dot{a}^2}{(1 - d\|\gamma\|_\infty)} \left| \frac{\partial \psi_n}{\partial s} \right|^2 + \frac{d^2 \dot{a}^2 \dot{\gamma}^2}{4(1 - d\|\gamma\|_\infty)^3} |\psi_n|^2 \right. \\ &\quad \left. + \frac{2(\dot{b}^2 + d^2 \ddot{a}^2)}{1 - d\|\gamma\|_\infty} \left| \frac{\partial \psi_n}{\partial u} \right|^2 + \frac{(\dot{b}^2 + d^2 \ddot{a}^2)}{2(1 - d\|\gamma\|_\infty)^3} |\psi_n|^2 \right) (1 + d\|\gamma\|_\infty) ds du. \end{aligned} \quad (3.20)$$

In view of (2.2) and (2.6) and the construction of ψ_n

$$\begin{aligned} \int_{\Omega} a_1^2 \left| \frac{\partial \varphi_n}{\partial x} \right|^2 dx dy &= \mathcal{O} \left(\|\tilde{a}_1\|_{L^\infty(\omega_n)}^2 \right) \int_{\Omega_0} \left(\left| \frac{\partial \psi_n}{\partial s} \right|^2 + \left| \frac{\partial \psi_n}{\partial u} \right|^2 + |\psi_n|^2 \right) ds du \\ &= \mathcal{O} \left(\|\tilde{a}_1\|_{L^\infty(\omega_n)}^2 \right). \end{aligned} \quad (3.21)$$

In the same way one can estimate the integrals

$$\int_{\Omega} a_2^2 \left| \frac{\partial \varphi_n}{\partial y} \right|^2 dx dy = \mathcal{O} \left(\|\tilde{a}_2\|_{L^\infty(\omega_n)}^2 \right).$$

For the remaining third and fourth integrals, we have

$$\begin{aligned} \int_{\Omega} \left| \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} \right|^2 |\varphi_n|^2 dx dy &= \mathcal{O} \left(\left\| \frac{\partial a_1}{\partial x} \right\|_{L^\infty(\omega_n)}^2 + \left\| \frac{\partial a_2}{\partial y} \right\|_{L^\infty(\omega_n)}^2 \right), \\ \int_{\Omega} (a_1^2 + a_2^2)^2 |\varphi_n|^2 dx dy &= \mathcal{O} \left(\|\tilde{a}_1\|_{L^\infty(\omega_n)}^4 + \|\tilde{a}_2\|_{L^\infty(\omega_n)}^4 \right). \end{aligned}$$

Combining the last four expressions, (3.19) and the inequality (3.17) one establishes our claim. $\square$

4. PROOF OF LEMMA 3.2

Let us denote $h(s, u) := f(s)g(s, u)$. Then

$$\begin{aligned}
& \int_{\Omega_0} |i\nabla h + \tilde{A}h|^2 ds du \\
&= \int_{\Omega_0} \left| if \frac{\partial g}{\partial s} + \tilde{a}_1 f g + if'g \right|^2 ds du + \int_{\Omega_0} \left| if \frac{\partial g}{\partial u} + \tilde{a}_2 f g \right|^2 ds du \\
&= \int_{\Omega_0} f^2 \left| i \frac{\partial g}{\partial s} + \tilde{a}_1 g \right|^2 ds du - i \int_{\Omega_0} \left(i \frac{\partial g}{\partial s} + \tilde{a}_1 g \right) f f' \bar{g} ds du \\
&\quad + i \int_{\Omega_0} \left(-i \frac{\partial \bar{g}}{\partial s} + \tilde{a}_1 \bar{g} \right) f f' g ds du + \int_{\Omega_0} f'^2 |g|^2 ds du + \int_{\Omega_0} f^2 \left| i \frac{\partial g}{\partial u} + \tilde{a}_2 g \right|^2 ds du \\
&= \int_{\Omega_0} f^2 \left| i \frac{\partial g}{\partial s} + \tilde{a}_1 g \right|^2 ds du + \int_{\Omega_0} \frac{\partial g}{\partial s} f f' \bar{g} ds du \\
&\quad + \int_{\Omega_0} \frac{\partial \bar{g}}{\partial s} f f' g ds du + \int_{\Omega_0} f'^2 |g|^2 ds du + \int_{\Omega_0} f^2 \left| i \frac{\partial g}{\partial u} + \tilde{a}_2 g \right|^2 ds du \\
&= \int_{\Omega_0} f^2 |i\nabla g + \tilde{A}g|^2 ds du + \int_{\Omega_0} \frac{\partial g}{\partial s} f f' \bar{g} ds du + \int_{\Omega_0} \frac{\partial \bar{g}}{\partial s} f f' g ds du + \int_{\Omega_0} f'^2 |g|^2 ds du.
\end{aligned} \tag{4.1}$$

One can check that

$$\int_{\Omega_0} \frac{\partial g}{\partial s} f f' \bar{g} ds du + \int_{\Omega_0} \frac{\partial \bar{g}}{\partial s} f f' g ds du = - \int_{\Omega_0} (f'^2 + f'') |g|^2 ds du.$$

Hence, the right-hand side of (4.1) becomes

$$\int_{\Omega_0} |i\nabla h + \tilde{A}h|^2 ds du = \int_{\Omega_0} f^2 |i\nabla g + \tilde{A}g|^2 ds du - \int_{\Omega_0} f'' |g|^2 ds du. \tag{4.2}$$

Now we are going to use the following Hardy-type inequality, see e.g. [6].

Theorem 4.1. *Let $B \in C^1(\mathbb{R}^2)$ be a real-valued magnetic field such that $B \neq 0$ in $D = \mathbb{R} \times (0, \pi)$. Then*

$$c_B \int_D \frac{|u|^2}{1+s^2} ds du \leq \int_D \left(|i\nabla u + Au|^2 - |u|^2 \right) ds du$$

holds for all $u \in \mathcal{H}_0^1(D)$, where A is a magnetic vector potential associated with B : $B = \text{rot}(A)$ and c_B is a positive constant.

Theorem 4.1 can be transformed as follows:

Theorem 4.2. *Let $B \in C^1(\mathbb{R}^2)$ be a real-valued magnetic field such that $B \not\equiv 0$ in $\Omega_0 = \mathbb{R} \times (0, d)$. Then*

$$\tilde{c}_B \int_{\Omega_0} \frac{|u|^2}{1+s^2} ds du \leq \int_{\Omega_0} \left(|i\nabla u + Au|^2 - \frac{\pi^2}{d^2} |u|^2 \right) ds du$$

holds for all $u \in \mathcal{H}_0^1(\Omega_0)$, where A is a magnetic vector potential associated with B : $B = \text{rot}(A)$ and the constant in the left-hand side is represented via the constant from the theorem above as follows:

$$\tilde{c}_B = c \frac{d^2}{\pi^2} B\left(\frac{sd}{\pi}, \frac{ud}{\pi}\right).$$

Employing above theorem at the left-hand side of (4.2) one obtains

$$\tilde{c}_{\tilde{B}} \int_{\Omega_0} \frac{|h|^2}{1+s^2} ds du + \frac{\pi^2}{d^2} \int_{\Omega_0} |h|^2 ds du \leq \int_{\Omega_0} f^2 |i\nabla g + \tilde{A}g|^2 ds du - \int_{\Omega_0} f'' |g|^2 ds du,$$

where $\tilde{B}$ is the magnetic field corresponding to the magnetic potential $\tilde{A}$: $\tilde{B} = \text{rot}(\tilde{A})$. This establishes the statement of Lemma 3.2 with the constant $C_{\tilde{A}} = \tilde{c}_{\tilde{B}}$.

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