

Dedicated to Professor Jan Stochel
on the occasion of his 70th birthday

ON THE MÖBIUS INVARIANT PRINCIPAL FUNCTIONS OF PINCUS

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Communicated by Alexander Gomilko

Abstract. In this semi-expository short note, we prove that the only homogeneous pure hyponormal operator T with $\text{rank}(T^*T - TT^*) = 1$, modulo unitary equivalence, is the unilateral shift.

Keywords: hyponormal operator, multiplicity, trace formula, homogeneous operators, principal function.

Mathematics Subject Classification: 47B20.

1. INTRODUCTION

In this paper, a Hilbert space $\mathcal{H}$ is assumed to be complex and separable and an operator T on $\mathcal{H}$ is assumed to be linear and bounded. The algebra of bounded linear operators on a $\mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$. An operator $A \in \mathcal{L}(\mathcal{H})$ is said to be *hyponormal* if $[A^*, A] := A^*A - AA^*$ is *non-negative*, that is, $\langle [A^*, A]f, f \rangle \geq 0$ for all $f \in \mathcal{H}$.

Let $\mathcal{H}$ be a Hilbert space and $\{e_n\}_{n \geq 0}$ be an orthonormal basis in $\mathcal{H}$. For any bounded non-negative operator B acting on $\mathcal{H}$, define its trace by setting

$$\text{tr}(B) = \sum_n \langle Be_n, e_n \rangle.$$

This definition of $\text{tr}(B)$ does not depend on the choice of the orthonormal basis that was chosen to define it.

An operator $A \in \mathcal{L}(\mathcal{H})$ is said to be in the *trace class* $\mathcal{S}_1(\mathcal{H})$ if $\sum_{n=0}^{\infty} \langle |A|e_n, e_n \rangle$ is finite. As usual, here $|A|$ is the unique positive square root of the self-adjoint operator A^*A .

The s -numbers $\{s_j(T)\}_{j=1}^{\infty}$ of a compact operator T are the eigenvalues of $(T^*T)^{\frac{1}{2}}$, counted with multiplicity and arranged in decreasing order. The trace norm is also given by the formula

$$\|T\|_1 = \sum_{j=1}^{\infty} s_j(T).$$

Let T be a trace class operator. Set

$$\Lambda_T := \{\lambda_j(T) : j = 1, 2, \dots, \nu(T)\}$$

be an enumeration of the non-zero eigenvalues of T counting multiplicities. The determinant of the operator $I + T$ is defined as follows:

$$\det(I + T) = \begin{cases} \prod_{j=1}^{\nu(T)} (1 + \lambda_j(T)), & \Lambda_T \neq \emptyset, \\ 1 & \Lambda_T = \emptyset. \end{cases}$$

In case $\nu(T)$ is infinite, the convergence of the product defining the determinant follows from the inequality $\sum_{j=1}^{\infty} |\lambda_j(T)| \leq \|T\|_1$, see [12, Chapter II]. We need the following crucial relationship between the trace and the determinant.

Recall the Jacobi formula for matrix exponential, namely, $\det \exp(B) = \exp(\operatorname{tr}(B))$. Now, suppose that T is a trace class operator with $\|T\|_1 < 1$. Then we define $\det(I + T) = \exp \operatorname{tr}(\log(I + T))$. Here $\log(I + T)$ is the logarithm of $I + T$ given by the series (convergent in the norm $\|\cdot\|_1$)

$$\log(I + T) = - \sum_{n=1}^{\infty} (-1)^n \frac{T^n}{n}, \quad (1.1)$$

see [9, p. 81].

Definition 1.1. A natural number m is said to be the (rational) multiplicity of an operator $T \in \mathcal{L}(\mathcal{H})$ if there exist vectors $\{x_i\}_{i \in I}$, for some indexing set I with $|I| = m$, such that

$$H = \bigvee \{f(T)x_i, i \in I, f \in \operatorname{Rat}(\sigma(T))\},$$

where $\operatorname{Rat}(\sigma(T))$ is the set of all rational functions r of the form $\frac{p}{q}$ for a pair of polynomials p and q with q not zero on $\sigma(T)$.

In this short note we study the class of hyponormal operators T with finite multiplicity. The remarkable inequality [5] of Berger and Shaw

$$\operatorname{tr}[T^*, T] \leq \frac{m}{\pi} \operatorname{Area}(\sigma(T)) \quad (1.2)$$

ensures that the self-commutator $[T^*, T]$ of such an operator is in the trace class. An immediate corollary is an inequality due to Putnam [23]: If $T \in \mathcal{L}(\mathcal{H})$ is hyponormal, then

$$\|[T^*, T]\| \leq \frac{1}{\pi} \operatorname{Area}(\sigma(T)). \quad (1.3)$$

The verification below of Putnam's inequality is taken from [16, Chapter VI, Theorem 2.1]. Pick a non-zero vector $x \in \mathcal{H}$ and set

$$\mathcal{H}_x := \bigvee \{f(T)x : f \in \operatorname{Rat}(\sigma(T))\}.$$

Let $T_x : \mathcal{H}_x \rightarrow \mathcal{H}_x$ be the restriction of the operator T to $\mathcal{H}_x$. The operator T_x is evidently hyponormal and it is rationally cyclic of multiplicity 1. We have

$$\begin{aligned} \langle [T^*, T]x, x \rangle &= \|Tx\|^2 - \|T^*x\|^2 \\ &\leq \|T_x x\|^2 - \|T_x^* x\|^2 \\ &= \langle [T_x^*, T_x]x, x \rangle \\ &\leq \operatorname{tr}([T_x^*, T_x]) \\ &\leq \frac{1}{\pi} \operatorname{Area}(\sigma(T_x)) \\ &\leq \frac{1}{\pi} \operatorname{Area}(\sigma(T)), \end{aligned}$$

where the penultimate inequality follows from Berger-Shaw inequality (1.2) and the last inequality is a consequence of the spectral inclusion $\sigma(T_x) \subseteq \sigma(T)$.

Remark 1.2. Among many consequences of Putnam's inequality, we single out one that we will need in what follows, namely, if T is a *pure* hyponormal operator, then $\operatorname{Area}(\sigma(T)) > 0$.

Moreover, we note that the *determinantal formula* due to Carey and Pincus, discussed below, connects the *principal function* g_T of the operator T with the trace of $[T^*, T]$ using the Helton–Howe *trace formula*. For a recent account, one may consult the book [13].

Definition 1.3. The bi-holomorphic automorphism group $\operatorname{Möb}$ of the unit disc consists of rational functions φ of the form:

$$\varphi(z) = \beta \frac{z - a}{1 - \bar{a}z}, \quad \beta \in \mathbb{T}, a \in \mathbb{D},$$

where $\mathbb{T}$ and $\mathbb{D}$ denote the unit circle and the open unit disc, respectively.

For an operator T with the spectrum $\sigma(T)$ contained in the closed unit disc $\bar{\mathbb{D}}$, by the spectral mapping theorem, $0 \notin \sigma(I - \bar{a}T)$ for any $a \in \mathbb{D}$. Hence, the operator $I - \bar{a}T$, $a \in \mathbb{D}$, is invertible.

Definition 1.4. An operator T with $\sigma(T) \subseteq \bar{\mathbb{D}}$ is said to be *homogeneous* if the operator

$$\varphi(T) := \beta(T - a)(I - \bar{a}T)^{-1}, \quad \beta \in \mathbb{T}, a \in \mathbb{D},$$

is unitarily equivalent to T for all $\varphi \in \operatorname{Möb}$.

The problem of determining all the homogeneous normal operators, homogeneous contractions and homogeneous operators in the Cowen–Douglas class has been addressed in a series of papers [2, 4, 15] previously. One of the goals of this paper is to determine modulo unitary equivalence, all hyponormal operators T such that $[T^*, T]$ is in trace class that are homogeneous. This involves, among other things, finding a transformation rule for the principal function of an operator under the Möbius transformations.

2. PRELIMINARIES

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *hyponormal* if the self-commutator $[T^*, T] = T^*T - TT^*$ is non-negative definite. A hyponormal operator T is said to be a *pure* if there is no nontrivial reducing subspace for T on which it is normal. Every hyponormal operator T , modulo unitary equivalence, is of the form $T_p \oplus T_n$, where $T_p = T|_{\mathcal{H}_p}$, $T_n = T|_{\mathcal{H}_n}$ and $\mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_n$ such that T_p is pure and T_n is normal, see [16, Chapter II, Theorem 1.3].

Any operator $T \in \mathcal{L}(\mathcal{H})$ can be written in the form $T = A + iB$, where $A = \frac{T+T^*}{2}$ and $B := \frac{T-T^*}{2i}$ are self-adjoint. It follows that $[T^*, T] = 2i[A, B]$.

2.1. PRINCIPAL FUNCTION

The principal function of an operator T is defined by means of an auxiliary operator valued function E of two complex variables, called the determining function of T . The principal function g_T of T then appears by expressing the multiplicative determinant of the self-commutator, or the trace of the self-commutator $D := [T^*, T]$ as an integral. We recall that the determining function E is given by the formula

$$E(z, w) = I - 2iD^{\frac{1}{2}}(A - z)^{-1}(B - w)^{-1}D^{\frac{1}{2}}, \quad z, w \in \mathbb{C} \setminus \sigma(A) \times \sigma(B).$$

Pincus in [18, 20] proved the existence of a function $g(v, u) \geq 0$ such that

$$\det E(z, w) = \exp \left(\frac{1}{2\pi i} \iint_{\mathbb{C}} g(u, v) \frac{du}{u - z} \frac{dv}{v - w} \right), \quad z, w \in \mathbb{C} \setminus \sigma(A) \times \sigma(B). \quad (2.1)$$

The support of any almost everywhere determined version of $g(u, v)$ is said to be the “determining set” of the pair A, B , or equivalently that of the operator T . The essential closure of the determining set is denoted by $D(A, B)$. It is proved in [19] that $\sigma(T) = D(A, B)$. Thus, $\text{Supp}(g) \subseteq \sigma(T)$ and if T is pure, then $\text{Supp}(g) = \sigma(T)$, see also [9, p. 105].

Remark 2.1. For every integrable, compactly supported function g on $\mathbb{C}$, with $0 \leq g \leq 1$, there exists a pure semi-normal operator T , with $\text{rank}[T^*, T] = 1$ such that $[g] = [g_T]$ in $L^1(d\mu)$. The proof is in [7, Theorem 1], see also [21].

2.2. THE TRACIAL BI-LINEAR FORM

Let $\mathbb{C}[x, y]$ denote the algebra polynomials over the complex field in the two indeterminates x, y . Thus, any $p \in \mathbb{C}[x, y]$ is of the form

$$p(x, y) = \sum_{j,k=1}^m a_{i,j} x^j y^k, \quad a_{i,j} \in \mathbb{C}.$$

Let A, B be a pair of self adjoint operators in $\mathcal{L}(\mathcal{H})$ such that $\|[A, B]\|_1 < \infty$. Also, let $\mathbb{C}[A, B]$ be the algebra of operators generated by substituting A, B in place of the

commuting variables x, y of the polynomial $p \in \mathbb{C}[x, y]$. Thus, if X, Y is any pair of operators in $\mathbb{C}[A, B]$, then the operator

$$p(X, Y) = \sum_{j,k=1}^m a_{i,j} X^j Y^k$$

is well defined modulo operators of trace class. The tracial bi-linear form associated with the pair X, Y is

$$(p, q) = \text{tr } i [p(X, Y), q(X, Y)], \quad p, q \in \mathbb{C}[x, y].$$

An amazing formula discovered by Helton and Howe [14] for the tracial bilinear form is given below.

Theorem 2.2 (Helton–Howe). *Suppose that X, Y are a pair of operators such that $[X, Y]$ is in trace class. Then there exists a regular signed Borel measure μ with compact support in $\mathbb{C}$ such that for $p, q \in \mathbb{C}[x, y]$,*

$$(p, q) = \text{tr } i [p(X, Y), q(X, Y)] = \iint_{\mathbb{C}} J(p, q) d\mu,$$

where

$$J(p, q) = \frac{\partial p}{\partial x} \frac{\partial q}{\partial y} - \frac{\partial p}{\partial y} \frac{\partial q}{\partial x}.$$

Soon after the discovery of the Helton–Howe formula, Pincus established that the measure μ in the Helton–Howe formula is mutually absolutely continuous with respect to the area measure $dx dy$, that is, $d\mu = g_T(x, y) dx dy$, where g_T is the principal function of the operator $T = X + iY$.

2.3. UNITARY INVARIANTS

For z, w in a neighbourhood of infinity, the operator valued determining function $E(z, w)$ of an irreducible pure hyponormal operator T of trace class is a complete unitary invariant of T . The principal function g_T , on the other hand, is a unitary invariant in general but it is a complete invariant when the rank of $[T^*, T]$ is 1, see [18]. Indeed, we describe a large class of operators showing that the principal function need not be a complete invariant if we drop the requirement: rank of $[T^*, T]$ is 1. A different unitary invariant is in [10]. In what follows, we assume that the operator T is a irreducible hyponormal (hence, pure), and that rank of $[T^*, T] = 1$. Thus, we assume without loss of generality that $[T^*, T] = x \otimes x$ for some $x \in \mathcal{H}$, where $x \otimes x$ denotes the *non-negative definite* rank one operator $h \mapsto \langle h, x \rangle x$, $h \in \mathcal{H}$. In this case the multiplicative commutator and therefore, the determining function E of the operator T can be calculated explicitly as follows.

For any pair of complex numbers z, w not in the spectrum of T , the operators $(T^* - \bar{w})^{-1}$ and $(T - z)^{-1}$ exist and the multiplicative commutator

$$(T - z)(T^* - \bar{w})(T - z)^{-1}(T^* - \bar{w})^{-1},$$

is in the *determinant class*, that is, it is of the form $I + K$, where K is in trace class:

$$\begin{aligned} & (T - z)(T^* - \bar{w})(T - z)^{-1}(T^* - \bar{w})^{-1} \\ &= ((T^*T - x \otimes x - zT^* - \bar{w}T + z\bar{w})(T - z)^{-1}(T^* - \bar{w})^{-1}) \\ &= ((T^* - \bar{w})(T - z)(T - z)^{-1}(T^* - \bar{w})^{-1} - (x \otimes x)(T - z)^{-1}(T^* - \bar{w})^{-1}) \\ &= I - (x \otimes x)(T - z)^{-1}(T^* - \bar{w})^{-1} \\ &= I + K, \end{aligned}$$

where

$$K = -(x \otimes x)(T - z)^{-1}(T^* - \bar{w})^{-1}$$

is in trace class, and moreover,

$$\operatorname{tr} K = -\langle (T^* - \bar{w})^{-1}x, (T^* - \bar{z})^{-1}x \rangle.$$

Therefore,

$$\begin{aligned} \det(I - K) &= \exp(\operatorname{tr} \log(I - K)) \\ &= \exp\left(\operatorname{tr} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} (-K)^j\right) \\ &= \exp\left(\sum_{j=1}^{\infty} \frac{(-1)^{2j+1}}{j} \operatorname{tr}(K^j)\right) \\ &= \exp\left(\sum_{j=1}^{\infty} \frac{(-1)^{2j+1}}{j} (\operatorname{tr}(K))^j\right) \\ &= \exp\left(\log(1 - \operatorname{tr}(K))\right) \\ &= 1 - \operatorname{tr}(K) \\ &= 1 - \langle (T^* - \bar{w})^{-1}x, (T^* - \bar{z})^{-1}x \rangle. \end{aligned}$$

Consequently, combining with the formula (2.1), we have the equality

$$1 - \langle (T^* - \bar{w})^{-1}x, (T^* - \bar{z})^{-1}x \rangle = \exp\left(-\frac{1}{\pi} \int_{\mathbb{C}} \frac{g_T(\zeta)}{(\zeta - z)(\bar{\zeta} - \bar{w})} dA(\zeta)\right). \quad (2.2)$$

For a different approach to establishing this formula, see [8, Theorem 4.3].

2.4. AN EXAMPLE

Let S be the unilateral shift operator acting on the Hilbert space ℓ_2 of square summable complex sequences by the rule: $Se_k = e_{k+1}$, where $\{e_0, e_1, e_2, \dots\}$ is the standard basis of ℓ_2 . The self-commutator $[S^*, S] = e_0 \otimes e_0$. Since $(S^* - \bar{w}I)^{-1}e_0 = -\frac{1}{\bar{w}}e_0$, we have that

$$1 - \langle (S^* - \bar{w}I)^{-1}e_0, (S^* - \bar{z}I)^{-1}e_0 \rangle = 1 - \left\langle -\frac{1}{\bar{w}}e_0, -\frac{1}{\bar{z}}e_0 \right\rangle = 1 - \frac{1}{z\bar{w}}.$$

We claim that

$$\exp \left(-\frac{1}{\pi} \int_{\mathbb{D}} \frac{1}{(\zeta - z)(\bar{\zeta} - \bar{w})} dA(\zeta) \right) = 1 - \frac{1}{z\bar{w}}.$$

Taking $|\zeta| \leq 1$, and $|z|, |w| > 1$, and expanding $\frac{1}{\zeta - z}$ as well as $\frac{1}{\bar{\zeta} - \bar{w}}$ in a power series of $\frac{\zeta}{z}$ and $\frac{\bar{\zeta}}{\bar{w}}$, respectively, the claim is verified by integrating the product term by term. Thus the principal function of the unilateral shift S is the characteristic function $\mathbb{1}_{\mathbb{D}}$ of the closed unit disc $\mathbb{D}$.

Remark 2.3. Let $\sigma_{\text{ess}}(T)$ be the essential spectrum of an operator T . For $\lambda \in \mathbb{C} \setminus \sigma_{\text{ess}}(T)$, the principal function $g(\lambda) = -\text{ind}(T - \lambda)$, see [9, 5⁰, p. 105]. Consequently, the principal function g_S of the unilateral shift S is $\mathbb{1}_{\mathbb{D}}$.

3. THE ACTION OF THE MÖBIUS GROUP

The hyponormal operators share an important property with normal operators, namely, the spectral radius $\rho(T)$ of a hyponormal operator equals its norm $\|T\|$. However, unlike normal operators, if T is a *pure* hyponormal operators, then by Putnam's inequality, the area measure of spectrum $\sigma(T)$ must be positive.

3.1. INVARIANCE

It is not hard to verify that if T is hyponormal, then $\varphi(T)$ is also hyponormal for any φ in Möb, the biholomorphic automorphism group of the unit disc $\mathbb{D}$. We reproduce the proof below from [24, Lemma 1].

Proposition 3.1 (Stampfli). *If T is hyponormal, then $\varphi(T)$, φ in Möb, is also hyponormal.*

Proof. Any Möbius transformation is a composition of an affine transformation and an inversion of some other affine transformation. We have

$$[(aT + b)^*, aT + b] = |a|^2[T^*, T] \geq 0.$$

Therefore, to complete the proof, it is enough to verify that $[(T^*)^{-1}, T^{-1}]$ is hyponormal for an invertible operator T . By hypothesis, we have that

$$0 \leq T^{-1}(T^*T - TT^*)(T^*)^{-1} = T^{-1}T^*T(T^*)^{-1} - I.$$

If A is invertible and $A \geq I$, then $A^{-1} \leq I$. Therefore,

$$I - T^*T^{-1}(T^*)^{-1}T \geq 0.$$

Hence,

$$\begin{aligned} [(T^*)^{-1}, T^{-1}] &= ((T^*)^{-1}T^{-1} - T^{-1}(T^*)^{-1}) \\ &= (T^*)^{-1}(I - T^*T^{-1}(T^*)^{-1}T)T^{-1} \geq 0 \end{aligned}$$

completing the proof of the proposition. $\square$

We now re-write the formula for the tracial bi-linear form in complex co-ordinates and in slightly greater generality, see [16, Chapter X, Theorem 2.4, and Equation (12), p. 242].

Theorem 3.2 (Carey–Helton–Howe–Pincus). *Suppose that $T \in \mathcal{L}(\mathcal{H})$ is a hyponormal operator with $[T^*, T]$ is in the trace class $\mathcal{S}_1(\mathcal{H})$. Then for any pair of functions p, q in the Frechet Space $C^\infty(\sigma(T))$ of all smooth functions on $\sigma(T)$, we have the equality*

$$\text{tr}[p(T, T^*), q(T, T^*)] = \frac{1}{\pi} \int_{\sigma(T)} J(p, q) g_T d\mu,$$

where

$$J(p, q) := \frac{\partial p}{\partial \bar{z}} \frac{\partial q}{\partial z} - \frac{\partial p}{\partial z} \frac{\partial q}{\partial \bar{z}}.$$

The exact relationship of this formula to that of the Helton–Howe formula given earlier is obtained by choosing $X = \text{Re } T$, $Y = \text{Im } T$; and converting the complex derivatives in $J(p, q)$ to real ones by using transformation from complex to real coordinates:

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)$$

and vice-versa.

The proof of the lemma below follows directly from the Carey–Helton–Howe–Pincus formula.

Lemma 3.3. *Suppose that $T \in \mathcal{L}(\mathcal{H})$ is a hyponormal operator and $[T^*, T]$ is in the trace class $\mathcal{S}_1(\mathcal{H})$ and that $\sigma(T) \subseteq \mathbb{D}$. Then $[(T^* - \bar{\lambda})^{-1}, (T - \lambda)^{-1}]$ is also in $\mathcal{S}_1(\mathcal{H})$ for $\lambda \notin \sigma(T)$. In particular, $[\varphi(T)^*, \varphi(T)]$ is in $\mathcal{S}_1(\mathcal{H})$ for any $\varphi \in \text{Möb}$.*

Proof. Pick $p(\zeta, \bar{\zeta}) = \frac{1}{\bar{\zeta} - \lambda}$ and $q(\zeta, \bar{\zeta}) = \frac{1}{\zeta - \lambda}$. Then $J(p, q) = \frac{1}{|\zeta - \lambda|^4} < k$ for some $k > 0$ since $\lambda \notin \bar{\mathbb{D}}$. Therefore,

$$\begin{aligned} \operatorname{tr}[(T^* - \bar{\lambda})^{-1}, (T - \lambda)^{-1}] &= \operatorname{tr}[p(T, T^*), q(T, T^*)] \\ &= \frac{1}{\pi} \int_{\sigma(T)} \frac{1}{|\zeta - \lambda|^4} g_T d\mu \\ &\leq \frac{k}{\pi} \|g_T\|_{L^1(\sigma(T))} < \infty. \end{aligned}$$

Since affine transform of a trace class operator is again in trace class, the proof is complete. $\square$

We now compute the self commutator of the operator $\varphi(T)$. For this, we note that $\varphi(z) = \frac{z-a}{1-\bar{a}z} = -(\bar{a})^{-1} + c(z - \bar{a}^{-1})^{-1}$, where $c = \frac{a-\bar{a}^{-1}}{a}$. Then

$$\begin{aligned} [\varphi(T)^*, \varphi(T)] &= [-a^{-1}I + \bar{c}(T^* - a^{-1})^{-1}, -\bar{a}^{-1}I + c(T - \bar{a}^{-1})^{-1}] \\ &= (-a^{-1}I + \bar{c}(T^* - a^{-1})^{-1})(-\bar{a}^{-1}I + c(T - \bar{a}^{-1})^{-1}) \\ &\quad - (-\bar{a}^{-1}I + c(T - \bar{a}^{-1})^{-1})(-a^{-1}I + \bar{c}(T^* - a^{-1})^{-1}) \\ &= |c|^2((T^* - a^{-1})^{-1}(T - \bar{a}^{-1})^{-1} - (T - \bar{a}^{-1})^{-1}(T^* - a^{-1})^{-1}) \\ &= |c|^2((T - \bar{a}^{-1})(T^* - a^{-1}))^{-1}[(T^* - a^{-1}), (T - \bar{a}^{-1})] \\ &\quad ((T^* - a^{-1})(T - \bar{a}^{-1}))^{-1} \\ &= |c|^2((T - \bar{a}^{-1})(T^* - a^{-1}))^{-1}[T^*, T]((T^* - a^{-1})(T - \bar{a}^{-1}))^{-1}. \end{aligned}$$

The computation of $[\varphi(T)^*, \varphi(T)]$ facilitates the proof of the lemma below.

Lemma 3.4. *Suppose that $T \in \mathcal{L}(\mathcal{H})$ is a hyponormal operator and the rank of $[T^*, T]$ is 1 and that $\sigma(T) \subseteq \bar{\mathbb{D}}$. Then the rank of the self-commutator $[\varphi(T)^*, \varphi(T)]$ is also 1.*

Proof. For the proof, in view of the preceding discussion, it is enough to verify that whenever T is an invertible operator with $\operatorname{rank}[T^*, T] = 1$, the rank of $[T^{*-1}, T^{-1}]$ is also 1. By hypothesis, $[T^*, T] = x \otimes x$ for some vector x in $\mathcal{H}$. Hence,

$$\begin{aligned} T^{-1}T^*T(T^*)^{-1} &= I + T^{-1}(T^*T - TT^*)(T^*)^{-1} \\ &= I + T^{-1}(x \otimes x)(T^*)^{-1}. \end{aligned}$$

Taking inverses on both sides, we have

$$\begin{aligned} T^{-1}(T^*)^{-1} &= (T^*)^{-1}[I + T^{-1}(x \otimes x)(T^*)^{-1}]^{-1}T^{-1} \\ &= (T\{I + T^{-1}(x \otimes x)(T^*)^{-1}\}T^*)^{-1} \\ &= (TT^* + x \otimes x)^{-1}. \end{aligned}$$

Similarly,

$$(T^*)^{-1}T^{-1} = (T^*T - x \otimes x)^{-1}.$$

Therefore,

$$\begin{aligned}
 [(T^*)^{-1}, T^{-1}] &= (T^*)^{-1}T^{-1} - T^{-1}(T^*)^{-1} \\
 &= (T^*T - x \otimes x)^{-1} - (TT^* + x \otimes x)^{-1} \\
 &= (T^*T - x \otimes x)^{-1}(\{TT^* + x \otimes x\} - \{T^*T - x \otimes x\}) \\
 &\quad (TT^* + x \otimes x)^{-1} \\
 &= (T^*T - x \otimes x)^{-1}[x \otimes x](TT^* + x \otimes x)^{-1} \\
 &= (TT^*)^{-1}(x \otimes x)(T^*T)^{-1}.
 \end{aligned}$$

It follows that the self-commutator of T^{-1} is also of rank one completing the proof. $\square$

Remark 3.5. Combining Proposition 3.1 and Lemma 3.4, we conclude that the set of *pure* hyponormal operators with rank 1 self-commutator is left invariant under the action of the Möbius group. Similarly, combining Proposition 3.1, this time with Lemma 3.3, we see that the set of *pure* hyponormal operators T with $\|[T^*, T]\|_1$ finite is also left invariant under the action of Möb.

3.2. A CHANGE OF VARIABLE FORMULA FOR THE PRINCIPAL FUNCTION

A change of variable formula for the principal function appears in [9, pp. 106–107] and also in [16, p. 245]. However, for our purposes, we need a change of variable formula for the principal function in the form given below.

Proposition 3.6. *Let T be a pure hyponormal operator with trace class self-commutator and set $W := \varphi(T)$, φ in Möb. Assume that the spectrum of T is contained in the closed unit disc. Then the relationship between the two principal functions g_T and g_W is given by the change of variable formula*

$$g_W(\zeta) = g_T(\varphi^{-1}(\zeta)), \quad \zeta \in \sigma(W).$$

Proof. We have proved that W is a hyponormal operator with $\|[W^*, W]\|_1 < \infty$. We note that $\varphi(T)^* = \varphi^*(T^*)$, where $\varphi^*(z) = \overline{\varphi(\bar{z})}$. Setting

$$\tilde{p}(z, \bar{z}) := p(\varphi(z), \overline{\varphi(z)}) \quad \text{and} \quad \tilde{q}(z, \bar{z}) := q(\varphi(z), \overline{\varphi(z)}),$$

we have that

$$\text{tr}[p(\varphi(T), \varphi(T)^*), q(\varphi(T), \varphi(T)^*)] = \frac{1}{\pi} \int_{\sigma(\varphi(T))} J(p, q) g_{\varphi(T)}(\zeta) dA(\zeta).$$

On the other hand,

$$\begin{aligned}
 \text{tr}[p(\varphi(T), \varphi(T)^*), q(\varphi(T), \varphi(T)^*)] &= \text{tr}[\tilde{p}(T, T^*), \tilde{q}(T, T^*)] \\
 &= \frac{1}{\pi} \int_{\sigma(T)} J_{\zeta}(\tilde{p}, \tilde{q}) g_T(\zeta) dA(\zeta) \\
 &= \frac{1}{\pi} \int_{\sigma(\varphi(T))} J_{\eta}(p, q) g_T(\varphi^{-1}(\eta)) dA(\eta),
 \end{aligned}$$

where $\eta = \varphi(\zeta)$. By the chain rule, we have $\frac{\partial \tilde{p}}{\partial \zeta} = \frac{\partial \tilde{p}}{\partial \bar{\eta}} \frac{\partial \bar{\eta}}{\partial \zeta}$, and similarly $\frac{\partial \tilde{p}}{\partial \bar{\zeta}} = \frac{\partial \tilde{p}}{\partial \eta} \frac{\partial \eta}{\partial \bar{\zeta}}$. Thus, we have the equality

$$J_{\zeta}(\tilde{p}, \tilde{q}) = J_{\eta}(p, q) \left(\frac{\partial(\overline{\varphi(\zeta)})}{\partial \bar{\zeta}} \frac{\partial(\varphi(\zeta))}{\partial \zeta} \right).$$

Consequently,

$$\begin{aligned} dA(\eta) &= -\frac{1}{2i} d\eta \wedge d\bar{\eta} \\ &= -\frac{1}{2i} \left(\frac{\partial(\overline{\varphi(\zeta)})}{\partial \bar{\zeta}} \frac{\partial(\varphi(\zeta))}{\partial \zeta} \right) d\zeta \wedge d\bar{\zeta} \\ &= \left(\frac{\partial(\overline{\varphi(\zeta)})}{\partial \bar{\zeta}} \frac{\partial(\varphi(\zeta))}{\partial \zeta} \right) dA(\zeta). \end{aligned}$$

Hence,

$$J_{\zeta}(\tilde{p}, \tilde{q}) dA(\zeta) = J_{\eta}(p, q) dA(\eta).$$

Since p and q are arbitrary C^{∞} functions on $\sigma(T)$, we conclude that

$$g_{\varphi(T)}(\zeta) = g_T(\varphi^{-1}(\zeta))$$

completing the proof. $\square$

4. HOMOGENEOUS HYPONORMAL OPERATORS T WITH $\text{rank}[T^*, T] = 1$

We have already remarked that the principal function of a pure hyponormal operator in the trace class $\mathcal{S}_1(\mathcal{H})$ is not a complete unitary invariant for the operator T in general. However, it is not hard to see that it is a unitary invariant.

Proposition 4.1. *Let T be a pure hyponormal operator in $\mathcal{S}_1(\mathcal{H})$. If W is an operator unitarily equivalent to T , then the principal functions of W and T coincide.*

Proof. Let $W = UTU^*$ for some unitary operator U . The operator W is hyponormal and is in $\mathcal{S}_1(\mathcal{H})$. For any polynomial $p \in \mathbb{C}[z, \bar{z}]$, we have $p(W, W^*) = Up(T, T^*)U^*$. Hence, by the Helton–Howe formula, we find that

$$\begin{aligned} \frac{1}{\pi} \int_{\sigma(W)} J(p, q) g_W(\zeta) dA(\zeta) &= \text{tr}[p(W, W^*), q(W, W^*)] \\ &= \text{tr}[Up(T, T^*)U^*, Uq(T, T^*)U^*] \\ &= \text{tr}(U[p(T, T^*), q(T, T^*)]U^*) \\ &= \text{tr}[p(T, T^*), q(T, T^*)] \\ &= \frac{1}{\pi} \int_{\sigma(T)} J(p, q) g_T(\zeta) dA(\zeta) \end{aligned}$$

Since $\sigma(T) = \sigma(W)$, we have that

$$\frac{1}{\pi} \int_{\sigma(T)} J(p, q)(g_T - g_W)(\zeta) dA(\zeta) = 0$$

for p, q in $\mathbb{C}[x, y]$, and in consequence $g_W = g_T$. $\square$

Imposing the condition of homogeneity on a pure hyponormal operator T in $\mathcal{S}_1(\mathcal{H})$, we investigate what happens to the principal function g_T .

We begin with the simple observation that if T is a homogeneous operator, then by the spectral mapping theorem, the spectrum $\sigma(T)$ must be invariant under the action of the Möbius group. Consequently, $\sigma(T)$ has to be either the closed unit disc $\overline{\mathbb{D}}$, or the unit circle $\mathbb{T}$. However, if T is also a pure hyponormal operator, then as we have noted earlier, $\sigma(T)$ cannot be $\mathbb{T}$.

Proposition 4.2. *Suppose that T is a pure hyponormal homogeneous operator such that $[T^*, T]$ is in $\mathcal{S}_1(\mathcal{H})$. Then the principal function g_T is constant on the spectrum $\sigma(T)$.*

Proof. Since $\varphi(T)$ is unitarily equivalent to T , $\varphi \in \text{Möb}$, it follows that $g_T(z) = g_{\varphi(T)}(z)$. By the change of variable formula for the principal function, we have $g_{\varphi(T)}(z) = g_T(\varphi^{-1}(z))$. Combining these two equalities, we conclude that

$$g_T(z) = g_{\varphi(T)}(z) = g_T(\varphi^{-1}(z)), \quad (4.1)$$

for all $\varphi \in \text{Möb}$. For a fixed but arbitrary $z \in \mathbb{D}$, pick a Möbius transformation φ_z with the property: $\varphi_z(0) = z$. Using this φ_z in Equation (4.1), we have

$$g_T(z) = g_{\varphi_z(T)}(z) = g_T(\varphi_z^{-1}(z)) = g_T(0).$$

We therefore conclude that g_T must be a constant on $\sigma(T)$, $0 < g_T(0) \leq 1$. $\square$

We have now all the tools to prove the only new result of this short note. Let us recapitulate what we have proved so far. Assume that T is a pure homogeneous hyponormal operator with rank of $[T^*, T] = 1$. Then for such an operator T we must have that:

- (i) the spectrum $\sigma(T) = \overline{\mathbb{D}}$,
- (ii) the principal function g_T must be a constant, moreover, this constant value is in $(0, 1]$,
- (iii) if $T = S$ is the unilateral shift, then $g_S = \mathbb{1}_{\overline{\mathbb{D}}}$, see Example 2.4.

Finally, note that:

- (a) the unilateral shift S is a homogeneous, see [3, List 4.1(2)], pure hyponormal operator and $\text{rank}([S^*, S]) = 1$,
- (b) the principal function of a pure hyponormal operator T with rank-one self-commutator $x \otimes x$ is a complete unitary invariant of T .

The remarkable assertion of (b) is due to Pincus and is in [18].

Moreover, in the case of a homogeneous pure hyponormal operator T with $\text{rank}[T^*, T] = 1$, the spectrum $\sigma(T) = \mathbb{D}$. Since the support of g_T equals $\sigma(T)$, therefore to find homogeneous pure hyponormal operators T with $\text{rank}[T^*, T] = 1$, we have to check if the operator with $0 < g_T(0) \leq 1$ is homogeneous. From what is said so far, it follows that the unilateral shift S is the unique (modulo unitary equivalence) pure hyponormal operator with $\text{rank}([S^*, S]) = 1$ such that $g_S = \mathbb{1}_{\mathbb{D}}$.

Theorem 4.3. *The only homogeneous pure hyponormal operator T with rank of $[T^*, T] = 1$, modulo unitary equivalence, is the unilateral shift.*

Proof. In view of the discussion preceding the theorem, we have to show that there is no homogeneous pure hyponormal operator T with rank of $[T^*, T] = 1$ such that $g_T = c < 1$. Let us suppose to the contrary that there exists such an operator T with $g_T = c < 1$. In the determinant expansion formula (2.2), setting $g_T = c$, we have (as in Example 2.4):

$$\begin{aligned} 1 - \langle (T^* - \bar{w})^{-1}x, (T^* - \bar{z})^{-1}x \rangle &= \exp \left(- \frac{1}{\pi} \int_{\sigma(T)} \frac{g_T(\zeta) dA(\zeta)}{(\zeta - z)(\bar{z} - \bar{w})} \right) \\ &= \left(1 - \frac{1}{z\bar{w}} \right)^c. \end{aligned} \quad (4.2)$$

Putting $z = w$ in Equation (4.2) we have the equality

$$1 - \|(T^* - \bar{w})^{-1}x\|^2 = \left(1 - \frac{1}{|w|^2} \right)^c. \quad (4.3)$$

Since T is homogeneous and hyponormal, the spectrum $\sigma(T)$ can only be $\mathbb{D}$, the possibility of $\sigma(T) = \mathbb{T}$ is ruled out by Putnam's inequality. For a hyponormal operator, the spectral radius $\rho(T) = \|T\|$ and we conclude that that

$$\|(T - wI)^{-1}\| = \rho((T - wI)^{-1}) \leq \frac{1}{|w|}.$$

Since $[T^*, T] = x \otimes x$ for some $x \in \mathcal{H}$ by hypothesis, taking $p(z, \bar{z}) = \bar{z}$ and $q(z, \bar{z}) = z$ in the Helton–Howe formula we conclude that $\|x\| = \sqrt{c}$. Therefore,

$$\|(T^* - \bar{w})^{-1}x\| \leq \sqrt{c} \|(T^* - \bar{w})^{-1}\|,$$

and we conclude that

$$\begin{aligned} 1 - \|(T^* - \bar{w})^{-1}x\|^2 &\geq 1 - \|(T^* - \bar{w})^{-1}\|^2 \|x\|^2 \\ &= 1 - c \|(T^* - \bar{w})^{-1}\|^2 \\ &\geq 1 - \frac{c}{|w|^2}. \end{aligned} \quad (4.4)$$

Combining the equality (4.3) with the inequality (4.4), we have

$$\left(1 - \frac{c}{|w|^2} \right) \leq \left(1 - \frac{1}{|w|^2} \right)^c, \quad |w| > 1. \quad (4.5)$$

It is easy to verify that the inequality (4.5) is false unless $c = 1$ completing the proof. $\square$

We give an example of a class $\mathcal{T}$ consisting of unitarily inequivalent homogeneous hyponormal operators such that $[T^*, T] \in \mathcal{S}_1$, $\text{rank } [T^*, T] = \infty$ such that $g_{T_1} = g_{T_2}$ for every pair of operators $T_1, T_2 \in \mathcal{T}$. Also, see remark below Lemma 1 in [22, p. 252].

Let $\mathcal{T}$ be the set of weighted shift operator $\{T_\lambda : \lambda > 1\}$ with weight sequences $\{w_n(\lambda)\}_{n \geq 0}$, $w_n(\lambda) = \sqrt{\frac{n+1}{n+\lambda}}$. For $\lambda > 1$, the weight sequence $\{w_n(\lambda)\}$ is strictly increasing and hence T_λ is hyponormal. The operator T_λ is also pure and cyclic. Clearly, $[T_\lambda^*, T_\lambda]$ is a diagonal operator D_λ with

$$D_\lambda(0, 0) = w_0 \quad \text{and} \quad D_\lambda(i, i) = w_{i+1}^2(\lambda) - w_i^2(\lambda) \neq 0.$$

Thus, $\text{rank}[T^*, T] = \infty$. Moreover,

$$\text{tr}[T_\lambda^*, T_\lambda] = \sum_{i=0}^{\infty} (w_{i+1}^2(\lambda) - w_i^2(\lambda)) + w_0^2(\lambda) = 1.$$

For $\lambda_1 \neq \lambda_2$, the two operators T_{λ_1} and T_{λ_2} are unitarily inequivalent. But all these operators are homogeneous, see [3]. Therefore, the principal function g_{T_λ} is constant, say c , on $\mathbb{D}$. But then

$$1 = \text{tr}[T_\lambda^*, T_\lambda] = \frac{1}{\pi} \int_{\mathbb{D}} c \, dA(\zeta).$$

Thus, $c = 1$ and it follows that g_{T_λ} is identically 1 on $\mathbb{D}$ for all $\lambda > 1$.

4.1. OPEN PROBLEM

Find all the pure hyponormal operators T such that $[T^*, T]$ is in $\mathcal{S}_1(\mathcal{H})$ and that g_T is constant on $\sigma(T)$ modulo unitary equivalence.

Remark 4.4. In studying homogeneous contractions T assuming that both the defect indices of T are equal to 1, it was shown that the Sz.-Nagy–Foias characteristic function of T must be constant. This observation leads to a class of homogeneous bi-lateral shifts (all of them inequivalent among themselves), parametrized by $c > 0$, possessing a constant characteristic function, see [1, 11].

Similarly, homogeneous operators T in the Cowen-Douglas class $B_1(\mathbb{D})$ are determined by specifying the curvature $\lambda = -\mathcal{K}_T(0) > 0$ just at one point. From this, one infers that an operator T in $B_1(\mathbb{D})$ is homogeneous if and only if T is of the form T_λ^* , $\lambda > 0$, discussed above (see [17]).

The situation involving the hyponormal operators T with $\text{rank}[T^*, T] = 1$, appears to be very different. Here again, the unitary invariant g_T , under the assumption of homogeneity, is a constant function, say c , with $0 < c \leq 1$. But there is only one *homogeneous* hyponormal operator T with $[T^*, T] = x \otimes x$, namely, the unilateral shift corresponding to $c = 1$.

4.2. POSTSCRIPT

In a conversation with the second author, in the year 1983, Kevin Clancey had remarked that the only homogeneous *pure* hyponormal operator with rank 1 self-commutator might be the unilateral shift. We have verified this statement to be correct in this short note.

Acknowledgements

We are very grateful to K.B. Sinha for his generous help during the preparation of this manuscript. In particular, the proof of Lemma 3.4 evolved during discussions with him. We also express our gratitude to M. Putinar for patiently answering some of our questions and pointing to the paper [22]. We thank the anonymous referee for carefully reading the manuscript and making several useful suggestion that helped in improving the exposition.

The second author gratefully acknowledges the financial support from the Science and Engineering Research Board (SERB) in the form of a J C Bose National Fellowship.

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Received: April 17, 2023.

Revised: August 22, 2023.

Accepted: August 23, 2023.