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# FROM SET-VALUED DYNAMICAL PROCESSES TO FRACTALS

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**Abstract.** We present a general theory of topological semiattractors and attractors for set-valued semigroups. Our results extend and unify those previously obtained by Lasota and Myjak. In particular, we naturally generalize the concept of semifractals for the systems acting on Hausdorff topological spaces. The main tool in our analysis is the notion of topological (Kuratowski) limits. We especially focus on the forward asymptotic behavior of discrete set-valued processes generated by sequences of iterated function systems. In this context, we establish sufficient conditions for the existence of fractal-type limit sets.

**Keywords:** topological limit, lower semicontinuous multifunction, iterated function system, set-valued process, attractor.

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## 1. INTRODUCTION

Discrete dynamics are closely associated with the difference equation

$$x(n+1) = f(x(n)). \tag{1.1}$$

Starting with some initial state $x(0)$ in the phase space, we obtain the next state $x(1)$ by applying a transformation $f$ to $x(0)$. To obtain $x(2)$ we apply $f$ to $x(1)$, and so on. By induction, for every positive integer $n$ we have that

$$x(n) = f^n(x(0)).$$

If, at each time step, there are multiple possible future states, it is natural to consider a difference inclusion of the form

$$x(n+1) \in F(x(n)) \tag{1.2}$$

rather than the single-valued equation (1.1), where $F$ is a set-valued map acting on the phase space. By induction, this leads to

$$x(n) \in F^n(x(0))$$

for every positive integer $n$. In particular, if at each step it is possible to apply a map chosen from a given family (finite or infinite), this leads naturally to the notion of an iterated function system and to the dynamics governed by its associated Barnsley–Hutchinson set-valued map $F$ (see details in Section 2.3 below). Iterated function systems (commonly abbreviated as IFS) have been a subject of sustained mathematical interest over the past few decades, and the literature on IFSs is now extensive. These systems also possess a wide range of applications in various fields.

On the other hand, one can consider dynamics in which the transformation varies at each time step. This leads to a non-autonomous difference equation

$$x(n+1) = f_n(x(n)) \tag{1.3}$$

or, more generally, a non-autonomous difference inclusion

$$x(n+1) \in F_n(x(n)) \tag{1.4}$$

which generalize equations (1.1) and (1.2), respectively. In this setting, the iterative process is replaced by a composition of maps taken from a given sequence. Extending this idea further, one can replace the set of positive integers with an arbitrary directed set. This, in turn, leads to the notion of a general process (see Section 2.4 below).

The notion of a semiattractor (sometimes referred to as a semifractal) was introduced in the final decade of the 20th century by A. Lasota and J. Myjak as a natural generalization of classical compact attractors (see [12]). The theory of compact attractors has been well developed and extensively studied, beginning with the foundational work of J.E. Hutchinson and M.F. Barnsley (see, for example, the survey [1] and the bibliography therein). In [16], Lasota and Myjak observed that, in certain computer graphics applications involving algorithms based on iterated function systems (IFS) with probabilities, classical systems composed solely of strict contractions are often extended to include nonexpansive mappings. In such cases, the system may no longer admit a compact attractor. Nevertheless, the random process (a Markov chain) induced by the IFS may still converge in distribution to a stationary measure $\mu_*$, though the support *supp* $\mu_*$ is not necessarily compact.

A natural question arises: can the set *supp* $\mu_*$ be associated in some way with a fixed point (possibly non-compact) of the Hutchinson operator $\overline{F}$ (see the definition below), which governs the deterministic dynamics of the system? An initial answer, proposed in the aforementioned paper, introduces the concept of the so-called *nucleus* of the system. More precisely, suppose that a classical IFS

$$\mathcal{S} = \{S_\sigma : X \to X : \sigma \in \{1, \ldots, N\}\}$$

contains a subsystem

$$\mathcal{S}_0 = \{S_\sigma : X \to X : \sigma \in \{1, \ldots, N_0\}\}$$

for some $N_0 \leq N$, and that this subsystem $\mathcal{S}_0$ possesses a compact attractor $A_*$, referred to as the nucleus. Then the Hutchinson operator $\overline{F}$ associated with the system $\mathcal{S}$ has a fixed point $C$, which can be constructed as

$$C = \overline{\bigcup_{n=1}^{\infty} \overline{F}^n(A_*)}$$

(cf. Theorem 3.6 below). This set $C$ is called the *semitractor* or *semifractal* of the system $\mathcal{S}$.

Subsequent work by A. Lasota and J. Myjak yielded additional significant results. These studies culminated in the paper [17]. In that paper a general definition of the semiattractor for a lower semicontinuous set-valued function was introduced, and fundamental properties were presented (see Theorem 3.1 below). It contains many other valuable results as well. In particular, it was shown that for a classical IFS with probabilities, the deterministic semifractal $C$ coincides with the support of the limiting distribution $\mu_*$.

Note that for a general IFS, this correspondence may not hold. However, if an IFS with probabilities admits a (topological) semiattractor $C$, and the limiting distribution $\mu_*$ of the associated Markov chain exists, then it always holds that *supp* $\mu_* \subset C$ (see, for example, [14, Theorem 4.4]).

Our goal is to develop a possibly general theory of semiattractors, as well as various types of attractors. Our approach encompasses the classical case studied by A. Lasota and J. Myjak. It includes set-valued semiflows (indexed by subsets of the real line), as well as more general set-valued semigroups.

The results obtained are valid not only in metric spaces but also in Hausdorff topological spaces. Consequently, a natural extension of the notions of fractals and semifractals is introduced within this new framework. Applications of the results obtained here to systems acting on non-metrizable spaces will be the subject of future work.

We employ the framework of topological limits. This approach simplifies many calculations and also allows us to obtain more general results than those derived using the Hausdorff–Pompeiu metric.

The organization of the paper is as follows. Section 2 is devoted to preliminary notions and results. We develop the necessary framework of topological limits, lower semicontinuous set-valued functions, iterated function systems, and general set-valued processes. Several results are proved that are difficult to find in the existing literature in the desired level of generality.

In Section 3, we introduce a general notion of semiattractors for nets of set-valued maps. The properties established there generalize, in a natural way, those presented in [17] and [8].

Section 4 introduces various types of attractors. In particular, we consider local (forward) attractors for set-valued processes and study their properties. Our approach arises naturally from the general definition, yet it differs from those found in the existing literature (see, for example, [5–7, 22], and the references therein). The relationship between the notions of attractors proposed here and the concepts of forward and pullback non-autonomous attractors remains an open problem.

Finally, in Section 5, we study attractors (fractals) induced by certain discrete set-valued processes, specifically those generated by sequences of IFSs. The results obtained are related to those for so-called asymptotically autonomous set-valued processes (see [6, 7]). Since the development of existence results for such objects is either absent or very limited in the literature (see [19]), we plan to explore this topic in future work. It is also worth noting that existing theorems on the limits of non-autonomous sequences of transformations are often difficult (or even impossible) to apply in our framework (see, for example, [24]).

It is worth noting that in [10] the notion of so-called evolution semiattractors was introduced to describe the asymptotic pullback behavior of set-valued processes. These sets play a significant role as supports of strongly mixing evolution systems of measures for stochastic flows (see details therein). However, since such sets inherently possess a non-autonomous nature, they fall outside the scope of the present study.

## 2. PRELIMINARIES

### 2.1. TOPOLOGICAL LIMITS

In most of the notions and results presented in this section, we follow the monograph [4] (see also [2, 20]), although we adopt a more convenient notation.

Let $\mathcal{A}$ be a nonempty set, and let $\preceq$ be a partial order on $\mathcal{A}$ (that is, a relation that is reflexive, antisymmetric, and transitive). Assume that every pair of elements in $\mathcal{A}$ has an upper bound. Then the pair $(\mathcal{A}, \preceq)$ is called a *directed set*. When the context is clear, we will refer to a directed set simply as $\mathcal{A}$.

**Example 2.1.** From the perspective of dynamical systems, the most important examples of directed sets are subsets of the real line $\mathbb{R}$ equipped with the natural order $\leq$. Examples include $\mathbb{Z}$, $\mathbb{Q}$, and subsets with a lower bound such as $\mathbb{N}$, $\mathbb{N} \cup \{0\}$, $(0, \infty)$, $[0, \infty)$, $\mathbb{Q} \cap (0, \infty)$, and $\mathbb{Q} \cap [0, \infty)$, among others.

**Example 2.2.** Consider a directed set $(\mathcal{A}, \preceq)$, and define

$$\mathcal{A}^2_{\preceq} := \{(\beta, \alpha) \in \mathcal{A} \times \mathcal{A} : \; \alpha \preceq \beta\}. \tag{2.1}$$

It is clear that $\mathcal{A}^2_{\preceq}$ is a directed set with the induced dictionary order.

For $\alpha \in \mathcal{A}$, define

$$\mathcal{A}_{+\alpha} := \{\beta \in \mathcal{A} : \; \alpha \preceq \beta\}. \tag{2.2}$$

Clearly, each set $\mathcal{A}_{+\alpha}$, for $\alpha \in \mathcal{A}$, is a directed set with the restricted order.

Assume that $\mathcal{A}$ is a directed set and $X$ is a nonempty set. A *net* of elements of $X$ is denoted by $(x_\alpha)_{\alpha \in \mathcal{A}}$.

It is clear that if $\mathcal{A} = \mathbb{N}$ or $\mathcal{A} = \mathbb{N} \cup \{0\}$, or more generally, using the notion introduced in formula (2.2), if $\mathcal{A} = \mathbb{Z}_{+k}$ for some $k \in \mathbb{Z}$, then a net is a standard sequence. If $\mathcal{A} = \mathbb{Z}$, it is a two-sided sequence.

If $\mathcal{A}$ is a directed set and $\mathcal{B} \subset \mathcal{A}$ is a nonempty subset, then $\mathcal{B}$ is called:

- *terminal* if there exists $\alpha_0 \in \mathcal{A}$ such that $\beta \in \mathcal{B}$ whenever $\alpha_0 \preceq \beta$,
- *cofinal* if for every $\alpha \in \mathcal{A}$ there exists $\beta \in \mathcal{B}$ such that $\alpha \preceq \beta$.

In what follows, let $X$ be a topological space. For convenience, we understand a neighbourhood of a point $x \in X$ to be any open set $U \subset X$ such that $x \in U$.

Assume that $(x_\alpha)_{\alpha\in\mathcal{A}}$ is a net (of points in $X$). We say that a point $x \in X$ is:

- a *limit* of $(x_\alpha)_{\alpha\in\mathcal{A}}$, denoted by $x = \lim_{\alpha\in\mathcal{A}} x_\alpha$ or $x_\alpha \underset{\alpha\in\mathcal{A}}{\longrightarrow} x$, if for every neighbourhood $U_x$ of $x$, there exists a terminal set $\mathcal{B} \subset \mathcal{A}$ such that $x_\alpha \in U_x$ for all $\alpha \in \mathcal{B}$,
- a *cluster point* of $(x_\alpha)_{\alpha\in\mathcal{A}}$ if for every neighbourhood $U_x$ of $x$, there exists a cofinal set $\mathcal{B} \subset \mathcal{A}$ such that $x_\alpha \in U_x$ for all $\alpha \in \mathcal{B}$.

**Remark 2.3.** It is known that for the set $\mathbb{N}$ of all positive integers, a nonempty subset $K \subset \mathbb{N}$ is terminal if and only if its complement $\mathbb{N} \setminus K$ is finite. Moreover, $K \subset \mathbb{N}$ is cofinal if and only if it is infinite. Hence, in the case of standard countable sequences of points in a metric space, the definitions of limits and cluster points coincide with the standard ones. In such cases, neighbourhoods can be replaced by open balls.

**Remark 2.4.** If $X$ is a Hausdorff topological space, then the limit of a net $(x_\alpha)_{\alpha\in\mathcal{A}}$, whenever it exists, is unique.

A mapping
$$\mathcal{A} \ni \alpha \mapsto A_\alpha \subset X$$
is called a *net of sets*. For a net of sets, we use the notation $(A_\alpha)_{\alpha\in\mathcal{A}}$. If $\mathcal{A} = \mathbb{N}$, we refer to it as a *sequence of sets*, denoted by $(A_n)_{n\in\mathbb{N}}$.

If $\mathcal{B} \subset \mathcal{A}$ is a terminal set, then a net $(x_\beta)_{\beta\in\mathcal{B}}$ with $x_\beta \in A_\beta$ is called a *selection*, or a *net of points chosen* from a net of sets $(A_\alpha)_{\alpha\in\mathcal{A}}$. In this case, we say that $x_\alpha \in A_\alpha$ *eventually*.

If $\mathcal{B} \subset \mathcal{A}$ is cofinal and $x_\beta \in A_\beta$ for $\beta \in \mathcal{B}$, we say that $x_\alpha \in A_\alpha$ *frequently*.

For a net of sets $(A_\alpha)_{\alpha\in\mathcal{A}}$ in a topological space $X$, we define:

- the *upper limit* $\limsup_{\alpha\in\mathcal{A}} A_\alpha$ as follows: $x \in \limsup_{\alpha\in\mathcal{A}} A_\alpha$ if and only if for every neighbourhood $U_x$ of $x$, we have $A_\alpha \cap U_x \neq \emptyset$ for $\alpha$ in some cofinal subset of $\mathcal{A}$,
- the *lower limit* $\liminf_{\alpha\in\mathcal{A}} A_\alpha$ as follows: $x \in \liminf_{\alpha\in\mathcal{A}} A_\alpha$ if and only if for every neighbourhood $U_x$ of $x$, we have $A_\alpha \cap U_x \neq \emptyset$ for $\alpha$ in some terminal subset of $\mathcal{A}$.

**Remark 2.5.** In the case where $\mathcal{A} = \mathbb{N}$ and $X$ is a metric space, the definitions of the upper and lower limits of a sequence $(A_n)_{n\in\mathbb{N}}$ of sets $A_n \subset X$ read as follows:

$$x \in \limsup_{n\in\mathbb{N}} A_n \iff \forall_{\epsilon>0}\forall_{n_0\in\mathbb{N}}\exists_{n\geq n_0}\ A_n \cap B^o(x,\epsilon) \neq \emptyset,$$
$$x \in \liminf_{n\in\mathbb{N}} A_n \iff \forall_{\epsilon>0}\exists_{n_0\in\mathbb{N}}\forall_{n\geq n_0}\ A_n \cap B^o(x,\epsilon) \neq \emptyset,$$

where the symbol $B^o(x,\epsilon)$ denotes an open ball centered at $x$ with radius $\epsilon > 0$.

These definitions were introduced by K. Kuratowski (see [15]).

**Remark 2.6.** Even though the lower and upper limits of a net of sets may be empty, their uniqueness is guaranteed in Hausdorff spaces.

The following characterization holds.

**Proposition 2.7.** *Let $X$ be a Hausdorff topological space and let $(A_\alpha)_{\alpha\in\mathcal{A}}$ be a net of subsets of $X$. Then:*

(i) $\liminf\limits_{\alpha\in\mathcal{A}} A_\alpha = \bigcap\left\{\overline{\bigcup\limits_{\beta\in\mathcal{B}} A_\beta} : \mathcal{B} \text{ is a cofinal subset of } \mathcal{A}\right\}$,

(ii) $\limsup\limits_{\alpha\in\mathcal{A}} A_\alpha = \bigcap\left\{\overline{\bigcup\limits_{\beta\in\mathcal{B}} A_\beta} : \mathcal{B} \text{ is a terminal subset of } \mathcal{A}\right\}$.

**Corollary 2.8.** *The sets* $\liminf_{\alpha\in\mathcal{A}} A_\alpha$ *and* $\limsup_{\alpha\in\mathcal{A}} A_\alpha$ *are closed subsets of $X$.*

Observe that $\liminf_{\alpha\in\mathcal{A}} A_\alpha \subset \limsup_{\alpha\in\mathcal{A}} A_\alpha$. If the equality

$$\liminf_{\alpha\in\mathcal{A}} A_\alpha = \limsup_{\alpha\in\mathcal{A}} A_\alpha,$$

holds, then we say that the net $(A_\alpha)_{\alpha\in\mathcal{A}}$ is *topologically convergent*. This common limit is called its *topological limit* and is denoted by $\lim_{\alpha\in\mathcal{A}} A_\alpha$.

The following properties of topological limits follow immediately from the definitions. Let $(A_\alpha)_{\alpha\in\mathcal{A}}$ and $(B_\beta)_{\beta\in\mathcal{A}}$ be nets of subsets of a Hausdorff topological space $X$.

If $A_\alpha \subset B_\alpha$ for all $\alpha \in \mathcal{A}$, then

$$\liminf_{\alpha\in\mathcal{A}} A_\alpha \subset \liminf_{\alpha\in\mathcal{A}} B_\alpha \quad \text{and} \quad \limsup_{\alpha\in\mathcal{A}} A_\alpha \subset \limsup_{\alpha\in\mathcal{A}} B_\alpha.$$

In particular, if $B_\alpha = a \ \subset X$ for every $\alpha \in \mathcal{A}$, then

$$\liminf_{\alpha\in\mathcal{A}} A_\alpha \subset \overline{A} \quad \text{and} \quad \limsup_{\alpha\in\mathcal{A}} A_\alpha \subset \overline{A}.$$

Moreover,

$$\liminf_{\alpha\in\mathcal{A}} A_\alpha = \liminf_{\alpha\in\mathcal{A}} \overline{A_\alpha} \quad \text{and} \quad \limsup_{\alpha\in\mathcal{A}} A_\alpha = \limsup_{\alpha\in\mathcal{A}} \overline{A_\alpha}.$$

Observe that if $\mathcal{B} \subset \mathcal{A}$ is a terminal set, then the intersection of $\mathcal{B}$ with any terminal subset of $\mathcal{A}$ is a terminal subset of $\mathcal{A}$, and the intersection of $\mathcal{B}$ with any cofinal subset of $\mathcal{A}$ is a cofinal subset of $\mathcal{A}$. Hence, we infer that

$$\liminf_{\alpha\in\mathcal{B}} A_\alpha = \liminf_{\alpha\in\mathcal{A}} A_\alpha \quad \text{and} \quad \limsup_{\alpha\in\mathcal{B}} A_\alpha = \limsup_{\alpha\in\mathcal{A}} A_\alpha.$$

Proposition 2.7 implies that

$$\bigcap_{\alpha\in\mathcal{A}} \overline{A_\alpha} \subset \liminf_{\alpha\in\mathcal{A}} A_\alpha \subset \limsup_{\alpha\in\mathcal{A}} A_\alpha \subset \overline{\bigcup_{\alpha\in\mathcal{A}} A_\alpha}. \tag{2.3}$$

We present the following refined result concerning monotone nets of subsets.

**Proposition 2.9.** *Let $(A_\alpha)_{\alpha\in\mathcal{A}}$ be a net of subsets of a Hausdorff topological space $X$.*

(a) *If the net $(A_\alpha)_{\alpha\in\mathcal{A}}$ is increasing, i.e., $A_{\alpha_1} \subseteq A_{\alpha_2}$ whenever $\alpha_1 \preceq \alpha_2$, then it converges topologically, and*

$$\lim_{\alpha\in\mathcal{A}} A_\alpha = \overline{\bigcup_{\alpha\in\mathcal{A}} A_\alpha}.$$

(b) *If the net $(A_\alpha)_{\alpha\in\mathcal{A}}$ is decreasing, i.e., $A_{\alpha_1} \supseteq A_{\alpha_2}$ whenever $\alpha_1 \preceq \alpha_2$, then it converges topologically, and*

$$\lim_{\alpha\in\mathcal{A}} A_\alpha = \bigcap_{\alpha\in\mathcal{A}} \overline{A_\alpha}.$$

*Proof.* (a) Since (2.3) holds, it suffices to prove that

$$\bigcup_{\alpha\in\mathcal{A}} A_\alpha \subset \liminf_{\alpha\in\mathcal{A}} A_\alpha. \tag{2.4}$$

As the net $(A_\alpha)_{\alpha\in\mathcal{A}}$ is increasing, for every cofinal subset $\mathcal{B} \subset \mathcal{A}$ we have

$$\bigcup_{\beta\in\mathcal{B}} A_\beta \subset \bigcup_{\alpha\in\mathcal{A}} A_\alpha,$$

and consequently,

$$\overline{\bigcup_{\beta\in\mathcal{B}} A_\beta} \subset \overline{\bigcup_{\alpha\in\mathcal{A}} A_\alpha}.$$

From this, it follows that

$$\bigcap \left\{ \overline{\bigcup_{\beta\in\mathcal{B}} A_\beta} : \mathcal{B} \text{ is a cofinal subset of } \mathcal{A} \right\} = \overline{\bigcup_{\alpha\in\mathcal{A}} A_\alpha}.$$

By Theorem 2.7(i), the left-hand side represents the lower limit $\liminf_{\alpha\in\mathcal{A}} A_\alpha$, so the inclusion (2.4) holds. (b) Using (2.3), it suffices to show that

$$\limsup_{\alpha\in\mathcal{A}} A_\alpha \subset \bigcap_{\alpha\in\mathcal{A}} \overline{A_\alpha}. \tag{2.5}$$

Let $\mathcal{B} \subset \mathcal{A}$ be a terminal subset. Since $(A_\alpha)_{\alpha\in\mathcal{A}}$ is decreasing, the net of closures $(\overline{A_\alpha})_{\alpha\in\mathcal{A}}$ is also decreasing. Consider an increasing sequence $(\beta_n)_{n\in\mathbb{N}}$ in $\mathcal{B}$, i.e., $\beta_1 \preceq \beta_2 \preceq \ldots$ Define the sets

$$B_n := \overline{\bigcup_{\beta_n \preceq \beta,\, \beta\in\mathcal{B}} \overline{A_\beta}} \quad \text{for } n \in \mathbb{N},$$

which form a decreasing sequence. Observe that

$$\bigcap_{n\in\mathbb{N}} B_n = \bigcap_{\alpha\in\mathcal{A}} \overline{A_\alpha}. \tag{2.6}$$

The following inclusion then holds:

$$\bigcap\left\{\overline{\bigcup_{\gamma\in\Gamma} A_\gamma} : \Gamma \text{ is a terminal subset of } \mathcal{A}\right\} \subset \bigcap_{n\in\mathbb{N}} B_n.$$

By Theorem 2.7(ii), the left-hand side is the upper limit $\limsup_{\alpha\in\mathcal{A}} A_\alpha$. Therefore, from this and the identity (2.6), the desired inclusion (2.5) follows. □

We will use the subsequent characterization of lower limits.

**Proposition 2.10.** *Let $X$ is a topological Hausdorff space and $(A_\alpha)_{\alpha\in\mathcal{A}}$ is a net of subsets of $X$. Therefore, $x \in \liminf_{\alpha\in\mathcal{A}} A_\alpha$ if and only if there is a net $(x_\alpha)_{\alpha\in\mathcal{A}}$ chosen from $(A_\alpha)_{\alpha\in\mathcal{A}}$ convergent to $x$.*

*Proof.* Let $(x_\alpha)_{\alpha\in\mathcal{A}}$ be a net selected from the family $(A_\alpha)_{\alpha\in\mathcal{A}}$. That is, there exists a terminal subset $\mathcal{B} \subset \mathcal{A}$ such that $x_\alpha \in A_\alpha$ for all $\alpha \in \mathcal{B}$. Suppose that $x = \lim_{\alpha\in\mathcal{B}} x_\alpha$, and let $U_x$ be an arbitrary neighborhood of $x$. By the definition of convergence of a net, there exists $\alpha_0 \in \mathcal{B}$ such that for all $\alpha \in \mathcal{B}$ with $\alpha_0 \preceq \alpha$, we have $x_\alpha \in U_x$. Since $x_\alpha \in A_\alpha$ for such $\alpha$, it follows that $A_\alpha \cap U_x \neq \emptyset$ for all $\alpha \succeq \alpha_0$ in $\mathcal{B}$. This implies that

$$x \in \liminf_{\alpha\in\mathcal{B}} A_\alpha = \liminf_{\alpha\in\mathcal{A}} A_\alpha.$$

To prove the converse implication, assume that $x \in \liminf_{\alpha\in\mathcal{A}} A_\alpha$. Let $\mathcal{B} \subset \mathcal{A}$ be a terminal subset. Consider a decreasing family of neighborhoods $\{U_x^\beta : \beta \in \mathcal{B}\}$ of $x$, meaning that $U_x^{\beta_1} \supset U_x^{\beta_2}$ whenever $\beta_1 \preceq \beta_2$.

By the definition of the lower limit, for each $\beta \in \mathcal{B}$, there exists $\alpha(\beta) \in \mathcal{B}$ such that

$$A_{\alpha(\beta)} \cap U_x^\beta \neq \emptyset.$$

Choose a point $x_\beta \in A_{\alpha(\beta)} \cap U_x^\beta$ for each $\beta \in \mathcal{B}$. We claim that the net $(x_\beta)_{\beta\in\mathcal{B}}$ converges to $x$.

Indeed, let $V_x$ be any neighborhood of $x$. Since the family $\{U_x^\beta\}_{\beta\in\mathcal{B}}$ is a neighborhood base at $x$, there exists $\beta_0 \in \mathcal{B}$ such that $U_x^{\beta_0} \subset V_x$. Then for all $\beta_0 \preceq \beta$, we have $U_x^\beta \subset V_x$, and hence $x_\beta \in U_x^\beta \subset V_x$. Since $V_x$ was arbitrary, it follows that $x_\beta \to x$, as required. □

A similar characterization of upper limits can be formulated: the upper limit of a net of sets consists of all cluster points of nets selected from the original net of sets.

**Remark 2.11.** In [4, Proposition 2.2.5], it was shown that if the space $X$ satisfies the first axiom of countability, then sequences may be used instead of nets in the characterization of lower and upper topological limits.

## 2.2. LOWER SEMICONTINUOUS SET-VALUED FUNCTIONS

In general, set-valued functions can be classified as lower semicontinuous, upper semicontinuous, or continuous. In this paper, we focus on the lower semicontinuous case, since iterated function systems are always lower semicontinuous. The upper semicontinuous case is related to the closed graph property, which does not necessarily hold for iterated function systems consisting of infinitely many functions. A set-valued function is continuous if and only if it is both lower and upper semicontinuous.

Let $X, Y$ be nonempty sets. By a *set-valued function* or a *set-valued map* $F : X \multimap Y$ we mean any nonempty subset $F$ of the product $X \times Y$. A set of all elements $x \in X$ such that the set $F(x) = \{y \in X : \ (x, y) \in F\}$ is nonempty is called the *domain* of $F$.

Given set-valued function $F : X \multimap Y$ and a subset $A \subset X$ we define the *image* of $A$ as a set

$$F(A) := \bigcup_{x \in A} F(x).$$

If, in addition, $Z$ is a nonempty set and $F : X \multimap Y$, $G : Y \multimap Z$ are set-valued functions, we define the *composition* $G \circ F$ of $F$ and $G$ as a set-valued function $G \circ F : X \multimap Z$ given by $(G \circ F)(x) = G(F(x))$. In particular, for a set-valued function $F : X \multimap X$ and every $k \in \mathbb{N}$ we denote

$$F^{k+1} = F \circ F^k.$$

In what follows, we mainly deal with lower semicontinuous set-valued functions. There exist many equivalent definitions of lower semicontinuity for set-valued maps; here, we choose a simple one that suits our purposes. Let $X$ and $Y$ be topological spaces. A set-valued function $F : X \multimap Y$ is said to be *lower semicontinuous* (abbreviated as *l.s.c.*) if for every subset $B \subset X$,

$$F(\overline{B}) \subset \overline{F(B)}.$$

The following characterization of lower semicontinuity holds:

**Proposition 2.12.** *Assume that $X$ and $Y$ are Hausdorff topological spaces. The following conditions are equivalent:*

(i) *the set-valued map $F : X \multimap Y$ is l.s.c.,*
(ii) *for every net $(x_\alpha)_{\alpha \in \mathcal{A}}$ in $X$ and every $x \in X$, the condition*

$$\lim_{\alpha \in \mathcal{A}} x_\alpha = x$$

*implies*

$$F(x) \subset \liminf_{\alpha \in \mathcal{A}} F(x_\alpha).$$

If $X$ and $Y$ are topological spaces, then with a set-valued map $F : X \multimap Y$ one can associate a transformation $\overline{F} : 2^X \to 2^Y$ defined by

$$\overline{F}(A) = \overline{F(A)} \quad \text{for } A \subset X$$

which is called the *Hutchinson operator* induced by $F$.

### 2.3. ITERATED FUNCTION SYSTEMS

Assume that $\Sigma$ is a nonempty set (of indices) and $X$ is a topological space. A family

$$\mathcal{S} = \{S_\sigma : X \to X : \sigma \in \Sigma\}$$

of continuous transformations is called an *iterated function system* (or *IFS* for short).

With an IFS $\mathcal{S}$, we associate the *Barnsley–Hutchinson* set-valued map $F : X \multimap X$ defined by

$$F(x) := \{S_\sigma(x) : \sigma \in \Sigma\} \quad \text{for } x \in X.$$

The following well-known result plays a key role.

**Proposition 2.13.** *Given an IFS $\mathcal{S}$, its Barnsley–Hutchinson set-valued function $F : X \multimap X$ is l.s.c.*

### 2.4. $\phi$-CONTRACTIONS

Let us introduce an important class of transformations called $\phi$-contractions.

Assume that $(X, d)$ is a metric space. A transformation $S : X \to X$ is called a *$\phi$-contraction* if there exists a non-decreasing function

$$\phi : [0, \infty) \to [0, \infty)$$

satisfying $\phi(t) < t$ for all $t > 0$ such that

$$d(S(x), S(y)) \leq \phi(d(x, y)) \quad \text{for all } x, y \in X. \tag{2.7}$$

**Proposition 2.14.** *Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of $\phi$-contractions (for the same function $\phi$). The following conditions are equivalent:*

(i) *the limit* $\lim_{n\to\infty} S_n \circ \ldots \circ S_1(x)$ *exists for some $x \in X$,*
(ii) *the limit* $\lim_{n\to\infty} S_n \circ \ldots \circ S_1(x)$ *exists for every $x \in X$ and does not depend on the choice of $x$.*

*Proof.* It suffices to prove that (i) implies (ii). Assume that

$$\lim_{n\to\infty} S_n \circ \ldots \circ S_1(x)$$

exists for some $x \in X$. Fix an arbitrary $y \in X$. Then, by property (2.7), we have

$$d\big(S_n \circ \ldots \circ S_1(x),\, S_n \circ \ldots \circ S_1(y)\big) \leq \phi^n(d(x, y))$$

for every $n \in \mathbb{N}$. Since $\phi^n(d(x, y)) \to 0$ as $n \to \infty$, it follows that the sequence

$$(S_n \circ \ldots \circ S_1(y))_{n\in\mathbb{N}}$$

converges, and its limit equals

$$\lim_{n\to\infty} S_n \circ \ldots \circ S_1(x).$$

As $y \in X$ was arbitrary, this proves that the limit exists for all $x \in X$ and is independent of the choice of $x$. □

The following result holds. The first assertion is due to [3], and the second one is a consequence of [13, Theorem 2].

**Proposition 2.15.** *Let $X$ be a complete metric space.*

(i) *If $S : X \to X$ is a $\phi$-contraction, then $S$ satisfies the conclusion of the Banach fixed point theorem, that is, $S$ has a unique fixed point $x_* \in X$ such that $S^n(x) \to x_*$ as $n \to \infty$ for every $x \in X$.*

(ii) *Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of $\phi$-contractions (for the same function $\phi$) which is pointwise convergent to a function $S : X \to X$. Then $S$ is a $\phi$-contraction. If also*

$$\liminf_{t\to\infty}(t - \phi(t)) > 0, \tag{2.8}$$

*then the sequence of fixed points of the functions $S_n$ converges to the fixed point of $S$.*

## 2.5. SET-VALUED PROCESSES

Assume now that $(\mathcal{A}, \preceq)$ is a directed set and the set $\mathcal{A}^2_{\preceq}$ is given by (2.1). Moreover, let $X$ be a topological space. Consider the family $\mathcal{F} = \{F_{\beta,\alpha} : X \multimap X : (\beta, \alpha) \in \mathcal{A}^2_{\preceq}\}$. $\mathcal{F}$ is said to be a *set-valued process* if

$$F_{\gamma,\beta} \circ F_{\beta,\alpha} = F_{\gamma,\alpha} \quad \text{for } \alpha \preceq \beta \preceq \gamma$$

and

$$F_{\alpha,\alpha}(x) = \{x\} \quad \text{for } \alpha \in \mathcal{A} \text{ and } x \in X.$$

**Example 2.16.** It is well known (see, for example, [10]) that in the case $\mathcal{A} = \mathbb{Z}$, the family $\mathcal{F} = \{F_{m,n} : X \multimap X : (m, n) \in \mathbb{Z}^2_{\geq}\}$ is a set-valued process (now called *discrete*) if and only if there exists a family $\mathcal{G} = \{G_n : X \multimap X : n \in \mathbb{Z}\}$ such that the following condition holds:

$$F_{m,n} = G_{m-1} \circ \ldots \circ G_n \quad \text{for} \quad n \leq m. \tag{2.9}$$

In particular, $F_{n+1,n} = G_n$ for every $n \in \mathbb{Z}$.

Now, for $n = 1$ we get

$$F_{m,1} = G_{m-1} \circ \ldots \circ G_1 \quad \text{for} \quad m \in \mathbb{N}.$$

It is easy to see that the dynamics of such a restriction lead us to the inclusion (1.4), and, in a particular case, if $G_m = G$ for every $m \in \mathbb{N}$, to the inclusion (1.2).

A family $\mathcal{G}$ is said to be *generating* a discrete set-valued process.

# 3. SEMIATTRACTORS

In [17], A. Lasota and J. Myjak introduced the following definition. Assume that $(X, d)$ is a metric space and $F : X \multimap X$ is a l.s.c. set-valued function. Denote

$$C := \bigcap_{x\in X} \liminf_{n\in\mathbb{N}} F^n(x).$$

If $C \neq \emptyset$, then it is called the *semiattractor* of $F$. Note that if a semiattractor exists, then it is unique (so we refer to it as *the* semiattractor) and it is a closed set.

In the cited paper, the following properties of semiattractors were proved.

**Proposition 3.1.** *Let $C$ be the semiattractor of a l.s.c. set-valued function $F : X \multimap X$. Then:*

(i) *if $A$ is a nonempty closed subset of $X$ such that $F(A) \subset A$, then $C \subset A$,*
(ii) $\overline{F(C)} = C$,
(iii) $C = \lim_{n\in\mathbb{N}} F^n(A)$ *for every nonempty set $A \subset C$.*

**Remark 3.2.** Associated with a set-valued function $F : X \multimap X$ is its Hutchinson operator $\overline{F} : 2^X \to 2^X$, defined by $\overline{F}(A) = \overline{F(A)}$ for every $A \subset X$.

Note that if the set-valued function $F$ has a semiattractor $C$, then, by Theorem 3.1(ii), $C$ is a fixed point of the operator $\overline{F}$. That is,

$$\overline{F}(C) = C.$$

**Remark 3.3.** Part (ii) of Theorem 3.1 can be interpreted as a property of *self-similarity* of the set $C$, which is why semiattractors are sometimes referred to as *semifractals*.

Property (iii) means that the semiattractor can be "recreated" from any of its fragments. This is often described as a *self-regeneration* property.

We propose the following generalization of the notion of a semiattractor for a net of set-valued functions defined on a topological space.

Here and in what follows, we assume that $(\mathcal{A}, \preceq)$ is a directed set and that $(F_\alpha)_{\alpha\in\mathcal{A}}$ is a net of set-valued functions $F_\alpha : X \multimap X$, where $\alpha \in \mathcal{A}$ and $X$ is a Hausdorff topological space. This general assumption will not be repeated.

If the set

$$C := \bigcap_{x\in X} \liminf_{\alpha\in\mathcal{A}} F_\alpha(x)$$

is nonempty, we call it the *semiattractor* of the net $(F_\alpha)_{\alpha\in\mathcal{A}}$. Clearly, if the semiattractor exists, it is unique and closed.

It seems, however, that without additional assumptions on the set-valued mappings $F_\alpha$, $\alpha \in \mathcal{A}$, as well as on the directed set $\mathcal{A}$ itself, it is impossible to establish any meaningful properties of the set $C$. Let us consider the following conditions:

(H1) Let $+ : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be an associative and commutative operation. Assume that it is consistent with the order on $\mathcal{A}$, that is,

$$\alpha_1 \preceq \alpha_2 \quad \Rightarrow \quad \alpha_1 + \beta \preceq \alpha_2 + \beta \quad \text{for all } \alpha_1, \alpha_2, \beta \in \mathcal{A}.$$

Assume that $(F_\alpha)_{\alpha\in\mathcal{A}}$ forms a semigroup, which means that the following translation equation is satisfied:

$$F_{\alpha+\beta} = F_\beta \circ F_\alpha \quad \text{for all } \alpha, \beta \in \mathcal{A}.$$

(H2) $F_\alpha$ is l.s.c. for every $\alpha \in \mathcal{A}$.

We are now in a position to prove the main result of this section, which serves as a generalized counterpart of Theorem 3.1.

**Theorem 3.4.** *If $C$ is the semiattractor of a net $(F_\alpha)_{\alpha\in\mathcal{A}}$, then:*

(a) *if $A \subset X$ is a closed, nonempty subset such that $F_\alpha(A) \subset A$ for every $\alpha \in \mathcal{A}$, then*

$$C \subset A.$$

*If, in addition,*

$$F_\alpha(C) \subset C \quad \textit{for all } \alpha \in \mathcal{A}, \tag{3.1}$$

*then:*

(b) *for every nonempty set $A \subset C$, we have*

$$C = \lim_{\alpha\in\mathcal{A}} F_\alpha(A),$$

(c) *if condition* (H1) *is satisfied, then*

$$\overline{F_\alpha(C)} = C \quad \textit{for every } \alpha \in \mathcal{A}.$$

*Proof.* (a) Let $A \neq \emptyset$ be a closed set satisfying $F_\alpha(A) \subset A$ for all $\alpha \in \mathcal{A}$, and let $x \in A$. Then, by the properties of lower and upper topological limits,

$$C \subset \liminf_{\alpha\in\mathcal{A}} F_\alpha(x) \subset \limsup_{\alpha\in\mathcal{A}} F_\alpha(x) \subset \limsup_{\alpha\in\mathcal{A}} F_\alpha(A) \subset \overline{A} = A.$$

Now suppose that the inclusion (3.1) holds. To prove (b), assume that $\emptyset \neq a \subset C$ and let $x \in A$. Then

$$C \subset \liminf_{\alpha\in\mathcal{A}} F_\alpha(x) \subset \liminf_{\alpha\in\mathcal{A}} F_\alpha(A) \subset \liminf_{\alpha\in\mathcal{A}} F_\alpha(C) \subset \limsup_{\alpha\in\mathcal{A}} F_\alpha(C) \subset C.$$

This implies that

$$C = \lim_{\alpha\in\mathcal{A}} F_\alpha(A).$$

Finally, to prove (c), assume that (H1) holds. Since the inclusion $F_\alpha(C) \subset C$ is satisfied for every $\alpha \in \mathcal{A}$, we have for any $\beta \in \mathcal{A}$ and $x \in C$ that

$$F_{\alpha+\beta}(x) \subset F_{\alpha+\beta}(C) \subset F_\alpha(F_\beta(C)) \subset F_\alpha(C).$$

Hence,

$$\liminf_{\alpha\in\mathcal{A}} F_\alpha(x) = \liminf_{\alpha\in\mathcal{A}} F_{\alpha+\beta}(x) \subset \overline{F_\alpha(C)}.$$

But since $C \subset \liminf_{\alpha\in\mathcal{A}} F_\alpha(x)$, it follows that $C \subset \overline{F_\alpha(C)}$. This inclusion, together with (3.1) and the closedness of $C$, yields the equality

$$\overline{F_\alpha(C)} = C \quad \text{for } \alpha \in \mathcal{A}.$$

This completes the proof. □

**Theorem 3.5.** *Assume that the set-valued maps $F_\alpha : X \multimap X$, $\alpha \in \mathcal{A}$, satisfy conditions* (H1) *and* (H2)*. Then*

$$F_\alpha(C) \subset C \quad \textit{for} \quad \alpha \in \mathcal{A},$$

*where $C$ is the semiattractor of the net $(F_\alpha)_{\alpha\in\mathcal{A}}$. In particular, for every nonempty set $A \subset C$, we have*

$$C = \lim_{\alpha\in\mathcal{A}} F_\alpha(A)$$

*and*

$$\overline{F_\alpha(C)} = C \quad \textit{for } \alpha \in \mathcal{A}.$$

*Proof.* Fix $\alpha_0 \in \mathcal{A}$ and let $y \in F_{\alpha_0}(C)$. Then there exists $x \in C$ such that $y \in F_{\alpha_0}(x)$. By the definition of a semiattractor, for every $z \in X$ we have $x \in \liminf_{\alpha\in\mathcal{A}} F_\alpha(z)$.

Therefore, using Theorem 2.10 on the characterization of the lower limit, we infer that there exists a terminal set $\mathcal{B} \subset \mathcal{A}$ and a net $(x_\alpha)_{\alpha\in\mathcal{B}}$ with $x_\alpha \in F_\alpha(z)$, converging to $x$. Since $F_{\alpha_0}$ is lower semicontinuous, we have

$$F_{\alpha_0}(x) \subset \liminf_{\alpha\in\mathcal{B}} F_{\alpha_0}(x_\alpha).$$

Hence,

$$F_{\alpha_0}(x) \subset \liminf_{\alpha\in\mathcal{B}} F_{\alpha_0}(F_\alpha(z)) = \liminf_{\alpha\in\mathcal{B}} F_{\alpha_0+\alpha}(z) = \liminf_{\alpha\in\mathcal{B}} F_\alpha(z) \subset \liminf_{\alpha\in\mathcal{A}} F_\alpha(z).$$

Since $z \in X$ was arbitrary, we conclude that

$$y \in F_{\alpha_0}(x) \subset C.$$

As $y \in F_{\alpha_0}(C)$ was arbitrary, it follows that

$$F_{\alpha_0}(C) \subset C.$$

□

**Theorem 3.6.** *Let $(F_\alpha)_{\alpha\in\mathcal{A}}$ be a net of set-valued functions $F_\alpha : X \multimap X$, $\alpha \in \mathcal{A}$, satisfying conditions* (H1) *and* (H2)*. Moreover, let $\mathcal{B} \subset \mathcal{A}$ be a terminal set, and let $(G_\alpha)_{\alpha\in\mathcal{B}}$ be a net of set-valued functions satisfying condition* (H1) *and*

$$G_\alpha(x) \subset F_\alpha(x) \quad \textit{for all } \alpha \in \mathcal{B} \textit{ and } x \in X. \tag{3.2}$$

*If $(G_\alpha)_{\alpha\in\mathcal{B}}$ has the semiattractor $C_G$, then $(F_\alpha)_{\alpha\in\mathcal{A}}$ has the semiattractor $C_F$, and*

$$C_G \subset C_F.$$

*Moreover,*

$$C_F = \lim_{\alpha\in\mathcal{A}} F_\alpha(C_G) = \overline{\bigcup_{\alpha\in\mathcal{A}} F_\alpha(C_G)}. \tag{3.3}$$

*Proof.* If $C_G$ is the semiattractor for $(G_\alpha)_{\alpha\in\mathcal{B}}$, then, using properties of topological limits, we obtain

$$C_G = \bigcap_{x\in X} \liminf_{\alpha\in\mathcal{B}} G_\alpha(x) \subset \bigcap_{x\in X} \liminf_{\alpha\in\mathcal{A}} F_\alpha(x) = C_F.$$

In particular, since $C_G \neq \emptyset$, it follows that $C_F \neq \emptyset$ as well, and (3.2) holds.

Since $C_G$ is a nonempty subset of $C_F$, by item (b) of Theorem 3.4, we immediately obtain

$$C_F = \lim_{\alpha\in\mathcal{A}} F_\alpha(C_G). \tag{3.4}$$

Moreover, since $F_\alpha(C_F) \subset C_F$ for all $\alpha \in \mathcal{A}$, we have

$$F_\alpha(C_G) \subset F_\alpha(C_F) \subset C_F \quad \text{for all } \alpha \in \mathcal{A}.$$

From this, and since $C_F$ is closed, it follows that

$$\overline{\bigcup_{\alpha\in\mathcal{A}} F_\alpha(C_G)} \subset C_F. \tag{3.5}$$

On the other hand, from the equality (3.4) and properties of topological limits, we have:

$$C_F = \limsup_{\alpha\in\mathcal{A}} F_\alpha(C_G) \subset \overline{\bigcup_{\alpha\in\mathcal{A}} F_\alpha(C_G)}.$$

In view of inclusion (3.5), the proof is complete. □

**Corollary 3.7.** *Assume that the net $(F_\alpha)_{\alpha\in\mathcal{A}}$ satisfies the assumptions of Theorem 3.6. Moreover, let $\mathcal{B} \subset \mathcal{A}$ be a terminal set, and let $(f_\alpha)_{\alpha\in\mathcal{B}}$ be a family of maps $f_\alpha : X \to X$, $\alpha \in \mathcal{B}$, which are selections of the set-valued functions $F_\alpha : X \multimap X$, that is,*

$$f_\alpha(x) \in F_\alpha(x) \quad \textit{for all } \alpha \in \mathcal{B} \textit{ and } x \in X.$$

*Assume moreover that for every $\alpha, \beta \in \mathcal{B}$ we have*

$$f_{\alpha+\beta} = f_\beta \circ f_\alpha.$$

*If there exists a point $x_* \in X$ such that*

$$f_\alpha(x_*) = x_* \quad \textit{for all } \alpha \in \mathcal{B},$$

*then the net $(F_\alpha)_{\alpha\in\mathcal{A}}$ has a semiattractor $C_F$, and*

$$C_F = \lim_{\alpha\in\mathcal{A}} F_\alpha(x_*) = \overline{\bigcup_{\alpha\in\mathcal{A}} F_\alpha(x_*)}.$$

*Proof.* For the proof, it is enough to observe that if we consider the induced set-valued maps $\{f_\alpha\} : X \multimap X$, defined by $\{f_\alpha\}(x) = \{f_\alpha(x)\}$ for $\alpha \in \mathcal{B}$, then the singleton set $\{x_*\}$ is the semiattractor of the generalized sequence $(\{f_\alpha\})_{\alpha\in\mathcal{B}}$.

Therefore, by applying Theorem 3.6, the result follows. □

**Remark 3.8.** The results presented in this section generalize not only those introduced by A. Lasota and J. Myjak in [17], as mentioned at the beginning, but also the results obtained in [8] by the first author in the framework of set-valued semiflows on metric spaces.

**Remark 3.9.** It is possible to formulate a local version of the results presented above. To do so, it suffices to consider, in the definition of the semiattractor, only points $x \in U$ for some nonempty open set $U \subset X$, and assume that $F_\alpha(U) \subset U$ for every $\alpha \in \mathcal{A}$, instead of considering all $x \in X$. This localized perspective is further explored in the next section.

## 4. LOCAL ATTRACTORS

In this section, we introduce the definition of a local attractor in a general framework. This concept extends the notion of an attractor for a single set-valued function on a metric space, originally introduced in [17].

As in the previous section, we assume that $X$ is a Hausdorff topological space, $(\mathcal{A}, \preceq)$ is a directed set and $(F_\alpha)_{\alpha\in\mathcal{A}}$ is a net of set-valued functions $F_\alpha : X \multimap X$, $\alpha \in \mathcal{A}$. Assume moreover that $U \subset X$ is a nonempty open set such that $F_\alpha(U) \subset U$ for every $\alpha \in \mathcal{A}$ and $\mathfrak{U} \subset 2^U$ is some class of nonempty subsets of $U$. We say that the set

$$A_{\mathfrak{U}} := \lim_{\alpha\in\mathcal{A}} F_\alpha(B)$$

is a (local) $\mathfrak{U}$-*attractor* of $(F_\alpha)_{\alpha\in\mathcal{A}}$ if the topological limit on the right-hand side exists, and it is independent of the choice of a set $B \in \mathfrak{U}$. In the case when $U = X$ such an attractor is called *global*.

Our definition covers some particular types of attractors appearing in the literature. In particular:

- if $\mathfrak{U}$ is the class of all singletons or, equivalently, the class of all finite subsets of $U$, then $A_{\mathfrak{U}}$ is said to be a *pointwise attractor*,
- if $\mathfrak{U}$ is the class of all compact subsets of $U$, and, additionally, for every compact set $K \subset U$ the image $F_\alpha(K)$ is also compact for every $\alpha \in \mathcal{A}$, then $A_{\mathfrak{U}}$ is said to be a *compact attractor*, whenever it is a compact set,
- if $X$ is a metric space and $\mathfrak{U}$ is the class of all bounded subsets of $X$, then $A_{\mathfrak{U}}$ is said to be a *Lasota–Myjak attractor*, or simply an *L-M attractor*.

We say that an IFS has a local semiattractor or an attractor if the sequence of iterates $(F^n)_{n\in\mathbb{N}}$ of its Barnsley–Hutchinson set-valued function $F$ possesses such a set.

**Remark 4.1.** Even for IFSs, all the aforementioned classes of attractors are substantially different. In [21], one can find examples of pointwise attractors that are not compact attractors. Moreover, in [11], it is shown that there exist unbounded Lasota–Myjak attractors. In particular, such L-M attractors cannot be compact. Examples of unbounded semiattractors were presented in [16, Example 6.2] and many others can be easily constructed by enriched of a contractive IFS with some non-contractive mappings (see, for example, [23]).

**Remark 4.2.** It is clear that every $\mathfrak{U}$-attractor is a (local) semiattractor. Hence, under the assumptions of Theorems 3.4 and 3.5, $\mathfrak{U}$-attractors are fixed points of the Hutchinson operators $\overline{F}_\alpha$, $\alpha \in \mathcal{A}$, associated with the set-valued functions $F_\alpha$, $\alpha \in \mathcal{A}$.

For general processes, the asymptotic behavior typically depends on the moment at which a trajectory starts. This is illustrated by the following example.

**Example 4.3.** Let $X$ be a nonempty set, $x_0 \in X$ a chosen element, and $A \subset X$ a nonempty subset such that $x_0 \notin A$. Consider the family $\mathcal{G} = \{G_n : X \multimap X : n \in \mathbb{Z}\}$ defined by

$$G_n(x) = \{x_0\} \quad \text{for } n < 0,$$

and

$$G_n(x) = \begin{cases} \{x_0\}, & \text{if } x = x_0, \\ A, & \text{if } x \neq x_0, \end{cases}$$

for every $x \in X$.

Fix $k \in \mathbb{Z}$ and consider the set $\mathbb{Z}_{+k} := \{k' \in \mathbb{Z} : k \leq k'\}$. Now observe that for the mappings $F_{m,n}$, $n \leq m$, generated by the family $\mathcal{G}$ via formula (2.9), we have

$$F_{m,n}(X) = \{x_0\} \cup A,$$

whenever $k \geq 0$, and

$$F_{m,n}(X) = \{x_0\},$$

whenever $k < 0$.

For this reason, we conclude that all "limit" sets in the case of significantly non-autonomous processes depend on the initial moment. For example, take $X = \mathbb{R}$, $x_0 = 0$, and $A = [1, 2]$. Then for any nonempty set $B \subset \mathbb{R}$, the topological limit

$$\lim_{m \in \mathbb{Z}_{+k}} F_{m,k}(B)$$

exists, is independent of $B$, and equals $\{0\}$ whenever $k < 0$. In the case when $k \geq 0$, this limit depends on the choice of $B$. Indeed:

- if $B = \{0\}$, then the limit is $\{0\}$,
- if $0 \notin B$, then the limit is $[1, 2]$,
- if $0 \in B$ and $B$ has at least two points, then the limit is $\{0\} \cup [1, 2]$.

Nevertheless, the following proposition, though perhaps unexpected, holds true.

**Proposition 4.4.** *Assume that* $\mathcal{F} = \{F_{\beta,\alpha} : X \multimap X \ : \ (\beta, \alpha) \in \mathcal{A}^2_{\preceq}\}$ *is a set-valued process on a Hausdorff topological space* $X$*,* $U \subset X$ *is a nonempty open set,* $\mathfrak{U} \subset 2^U$ *is a class of nonempty subsets of* $U$*, and* $F_{\beta,\alpha}(\mathfrak{U}) \subset \mathfrak{U}$ *for every* $(\beta, \alpha) \in \mathcal{A}^2_{\preceq}$*. If* $\alpha', \alpha'' \in \mathcal{A}$ *are arbitrary and the topological limits*

$$\lim_{\alpha \in \mathcal{A}_{+\alpha'}} F_{\alpha,\alpha'}(B) \quad \text{and} \quad \lim_{\alpha \in \mathcal{A}_{+\alpha''}} F_{\alpha,\alpha''}(B)$$

*exist and are independent of the choice of a set* $B \in \mathfrak{U}$*, then they are equal.*

*Proof.* Indeed, if $\alpha' \preceq \alpha''$, then for a set $B \in \mathfrak{U}$, we also have $B' := F_{\alpha'',\alpha'}(B) \in \mathfrak{U}$. Hence, by the assumption, the definition of a process, and the properties of topological limits,

$$\lim_{\alpha \in \mathcal{A}_{+\alpha'}} F_{\alpha,\alpha'}(B) = \lim_{\alpha \in \mathcal{A}_{+\alpha'}} F_{\alpha,\alpha''} \circ F_{\alpha'',\alpha'}(B) = \lim_{\alpha \in \mathcal{A}_{+\alpha''}} F_{\alpha,\alpha''}(B'). \qquad \square$$

This leads us to the definition of $\mathfrak{U}$-attractors for set-valued processes. Namely, let $\mathcal{F} = \{F_{\beta,\alpha} : X \multimap X \ : \ (\beta,\alpha) \in \mathcal{A}^2_{\preceq}\}$ be a set-valued process on a Hausdorff topological space $X$. Assume that $U \subset X$ is a nonempty open set, and let $\mathfrak{U} \subset 2^U$ be a class of nonempty subsets of $U$ such that $F_{\beta,\alpha}(\mathfrak{U}) \subset \mathfrak{U}$ for every $(\beta,\alpha) \in \mathcal{A}^2_{\preceq}$.

The set

$$A_{\mathfrak{U}} := \lim_{\beta \in \mathcal{A}_{+\alpha}} F_{\beta,\alpha}(B)$$

is called a (local) $\mathfrak{U}$-*attractor* of $\mathcal{F}$ if the topological limit on the right-hand side exists for some, or equivalently for all, $\alpha \in \mathcal{A}$ and is independent of the choice of a set $B \in \mathfrak{U}$.

**Remark 4.5.** In the particular case where $\mathcal{A} = \mathbb{Z}$, we know that

$$F_{m,n} = G_{m-1} \circ \ldots \circ G_n \quad \text{for every } m \geq n,$$

for some set-valued maps $G_n : X \multimap X$, $n \in \mathbb{Z}$. Since

$$F_{m+1,m} = G_m \quad \text{for every } m \in \mathbb{Z},$$

the condition $F_{m,n}(\mathfrak{U}) \subset \mathfrak{U}$ for every $(m,n) \in \mathbb{Z}^2_{\leq}$ is equivalent to $G_n(\mathfrak{U}) \subset \mathfrak{U}$ for every $n \in \mathbb{Z}$.

Therefore, by Proposition 4.4, a $\mathfrak{U}$-attractor $A_{\mathfrak{U}}$ of $\mathcal{F}$ exists if and only if the topological limit

$$\lim_{m \in \mathbb{N}} (G_m \circ \ldots \circ G_1)(B)$$

exists and is independent of the choice of $B \in \mathfrak{U}$. Hence, we define

$$A_{\mathfrak{U}} := \lim_{m \in \mathbb{N}} (G_m \circ \ldots \circ G_1)(B). \tag{4.1}$$

**Remark 4.6.** If now $\mathfrak{U} = \mathcal{H}(X)$ is the class of all nonempty bounded and closed subsets of a metric space $(X,d)$, then under the assumption that $G_n\big(\mathcal{H}(X)\big) \subset \mathcal{H}(X)$, we obtain that the global $\mathcal{H}(X)$-attractor can be expressed by the formula

$$A_{\mathcal{H}(X)} := \lim_{m \to \infty} (G_m \circ \ldots \circ G_1)(B), \tag{4.2}$$

for every $B \in \mathcal{H}(X)$, where the limit on the right-hand side is taken with respect to the Hausdorff–Pompeiu semimetric.

However, such a limit can be unbounded. For example, if we take $X = [0,\infty)$ and define $G_n(x) = [0,n]$ for every $x \in [0,\infty)$ and $n \in \mathbb{N}$, then the set-valued process generated by the family $\{G_n : n \in \mathbb{Z}\}$ has a $\mathcal{H}([0,\infty))$-attractor. Since the limit

$$\lim_{m \to \infty} (G_m \circ \ldots \circ G_1)(B) = [0,\infty)$$

does not depend on $B \in \mathcal{H}([0,\infty))$, we conclude that

$$A_{\mathcal{H}([0,\infty))} = [0,\infty).$$

## 5. FRACTALS FROM SEQUENCES OF IFSS

In this section, we consider sequences of iterated function systems (IFSs) indexed by a common index set. The corresponding set-valued process is generated by a sequence of associated Barnsley–Hutchinson set-valued functions. As mentioned in Remark 4.6, to study asymptotic behavior, it suffices to consider sequences indexed by $\mathbb{N}$.

As shown in [9], the following result holds.

**Proposition 5.1.** *Assume that $\Sigma$ is a nonempty set, $(X, d)$ is a complete metric space, and $S = \{S_\sigma : X \to X \ : \ \sigma \in \Sigma\}$ is an IFS consisting of $\phi$-contractions $S_\sigma$ with a function $\phi$ independent of $\sigma \in \Sigma$. Then $\mathcal{S}$ admits an L-M attractor.*

**Proposition 5.2.** *Assume that $\Sigma$ is a nonempty set and $(X, d)$ is a complete metric space. For every $n \in \mathbb{N}$, let $\mathcal{S}_n = \{S_{n,\sigma} : X \to X \ : \ \sigma \in \Sigma\}$ be an IFS consisting of $\phi$-contractions for the same function $\phi$ satisfying condition (2.8). If, for every $\sigma \in \Sigma$, the sequence $(S_{n,\sigma})_{n\in\mathbb{N}}$ converges pointwise to some transformation $S_\sigma : X \to X$, then the limit IFS $\mathcal{S} = \{S_\sigma : X \to X \ : \ \sigma \in \Sigma\}$ admits an L-M attractor.*

*Proof.* Assertion (ii) of Proposition 2.15 implies that for every $\sigma \in \Sigma$, the transformation $S_\sigma : X \to X$ is a $\phi$-contraction. Therefore, it is enough to apply Proposition 5.1 to the IFS $\mathcal{S} = \{S_\sigma : X \to X \ : \ \sigma \in \Sigma\}$. □

**Example 5.3.** Assume that $X = \mathbb{R}$, $\Sigma = \{0, 1\}$, and for every $n \in \mathbb{N}$ the transformations $S_{n,0}, S_{n,1} : \mathbb{R} \to \mathbb{R}$ are given by the formulas

$$S_{n,0}(x) = a_n \cdot x \quad \text{and} \quad S_{n,1}(x) = a_n \cdot x + (1 - a_n),$$

with some $a_n \in (0, 1)$ for every $n \in \mathbb{N}$.

Assume now that $a_n \to a \ \in (0, \frac{1}{2})$ as $n \to \infty$. Clearly, the sequences $(S_{n,0})_{n\in\mathbb{N}}$ and $(S_{n,1})_{n\in\mathbb{N}}$ converge pointwise to the transformations $S_0, S_1 : \mathbb{R} \to \mathbb{R}$ given by

$$S_0(x) = a \cdot x \quad \text{and} \quad S_1(x) = a \cdot x + (1 - a).$$

It is well known that the attractor of the IFS $\mathcal{S} = \{S_0, S_1\}$ is a Cantor set.

**Remark 5.4.** We see in example 5.3 that the L-M attractor of the limit IFS is not obtained as an attractor of any set-valued process generated by the sequence of associated Barnsley–Hutchinson set-valued functions. The relationship between these objects remains unclear and appears to require further investigation.

In [25] (see also the continuation of this research in [18]), certain evolutionary phenomena concerning compact attractors of classical IFSs were studied. More precisely, the authors considered the behavior of attractors of finite families of affine self-maps of the Euclidean space $\mathbb{R}^d$ under continuous variation of both the contractivity coefficients and the translation vectors.

Note that the approach presented above is different, as we require only pointwise convergence of sequences from a much broader class of weakly contractive mappings.

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REFERENCES

[1] M.F. Barnsley, A. Vince, *Developments in fractal geometry*, Bull. Math. Sci. **3** (2013), 299–348.

[2] G. Beer, *Topologies on Closed and Closed Convex Sets*, Springer Dordrecht, 1993.

[3] F. Browder, *On the convergence of successive approximations for nonlinear functional equations*, Indag. Math. **30** (1968), 27–35.

[4] R.S. Burachik, A.N. Iusem, *Set-Valued Mappings and Enlargements of Monotone Operators*, Springer, 2008.

[5] H. Cui, P.E. Kloeden, *Invariant forward attractors of non-autonomous random dynamical systems*, J. Differential Equ. **265** (2018), no. 12, 6166–6186.

[6] H. Cui, P.E. Kloeden, *Tail convergence of pullback attractors for asymptotically converging multi-valued dynamical system*, Asymptotic Anal. **112** (2019), no. 3–4, 165–184.

[7] H. Cui, P.E. Kloeden, *Comparison of attractors of asymptotically equivalent difference equations*, Springer Pros. Math. Stat. **287** (2019), 31–50.

[8] G. Guzik, *Semiattractors of set-valued semiflows*, J. Math. Anal. Appl, **435** (2016), no. 2, 1321–1334.

[9] G. Guzik, *On a class of cocycles having attractors which consists of singletons*, Topol. Methods Nonlinear Anal. **50** (2017), no. 2, 727–739.

[10] G. Guzik, *On supports of evolution systems of measures for converging in law non-homogenous Markov processes*, Topol. Methods Nonlinear Anal. **55** (2020), no. 1, 19–36.

[11] G. Guzik, *Boundedness of Lasota–Myjak attractors for iterated function systems*, submitted.

[12] J.E. Hutchinson, *Fractals and self similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.

[13] J. Jachymski, I. Jóźwik, *Nonlinear contractive conditions: a comparison and related problems*, Banach Center Publ. **77** (2007), 123–146

[14] R. Kapica, J. Morawiec, *Refinement equations and Feller operators*, Integr. Equ. Oper. Theory **70** (2011), 323–331.

[15] K. Kuratowski, *Wstęp do teorii mnogości i topologii*, PWN, Warszawa, 1962 [in Polish].

[16] A. Lasota, J. Myjak, *Semifractals*, Bull. Pol. Acad. Sci. Math. **44** (1996), no. 1, 5–21.

[17] A. Lasota, J. Myjak, *Attractors of multifunctions*, Bull. Pol. Acad. Sci. Math. **48** (2000), no. 3, 319–334.

[18] K. Leśniak, N. Snigireva, F. Strobin, A Vince, *Transition phenomena for the attractor of an iterated function system*, Nonlinearity **35** (2022), no. 10, 5396–5426.

[19] G. Manjunath, H. Jaeger, *The dynamics of random difference equations is remodeled by closed relations*, SIAM J. Math. Anal. **46** (2014), no. 1, 459–483.

[20] S. Mrówka, *On the convergence of nets of sets*, Fundamenta Math. **45** (1958), 247–253.

[21] M. Nowak, *Pointwise attractors which are not strict*, Ind. Math. **35** (2024), no. 1, 119–130.

[22] L. Rempe-Gillen, M. Urbański, *Non-autonomous conformal iterated function systems and Moran-set constructions*, Trans. Amer. Math. Soc. **368** (2016), no. 3, 1979–2017.

[23] F. Strobin, *Contractive iterated function systems enriched with nonexpansive maps*, Results Math. **76** (2021), Article no. 153.

[24] M. Tyran-Kamińska, *Stability of Markov processes nonhomogenous in time*, Ann. Pol. Math. **71** (1999), no. 1, 47–59.

[25] A. Vince, *Thresholds for one-parameter families of affine iterated function systems*, Nonlinearity **33** (2020), no. 12, 6541–6563.

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