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PhD THESIS

SUFFICIENT CONDITIONS FOR EXISTENCE OF LONG CYCLES IN GRAPHS

LECH ADAMUS

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Contents

Chapter 1

Polish summary

Wybrane warunki wystarczające istnienia długich cykli w grafach

Tematem rozprawy doktorskiej jest poszukiwanie warunków wystarczających na istnienie długich cykli w grafach, czyli cykli przechodzących przez ponad połowę wierzchołków w danym grafie. Celem jest określenie minimalnego rozmiaru grafu zapewniającego zawieranie się cyklu danej długości, ewentualnie określenie tego rozmiaru jednocześnie z minimalnym stopniem grafu. Problem będzie rozważany dla grafów prostych, w szczególności dwudzielnych, jak i dla grafów skierowanych (tam interesujące będą cykle o dowolnej orientacji).

Przy wszystkich twierdzeniach prezentowanych w tej rozprawie widnieją nazwiska ich autorów oraz podany jest odnośnik do literatury, gdzie dany wynik został opublikowany. Inicjały LA oznaczają, że autorem lub współautorem danego rezultatu jest piszący tę rozprawę. Twierdzenia takie są podane w pracy wraz z pełnymi ich dowodami.

Podstawowe pojęcia i terminologia używane w tej pracy są zgodne z [20, Chapter I]. Notacja stosowana w rozprawie jest przedstawiona w rozdziale 4.1. W szczególności, dla grafu G oznaczamy jego rząd, czyli ilość jego wierzchołków, przez $|G|$, natomiast rozmiar G, czyli ilość jego krawędzi, będzie oznaczany przez $||G||$. Dla danego wierzchołka x w grafie G zbiór wierzchołków z nim sąsiadujących w G będziemy oznaczali przez $N_G(x)$, $d_G(x)$ to stopień wierzchołka x w G (tzn. $d_G(x) = |N_G(x)|$), natomiast $\delta(G)$ to minimalny stopień grafu G. Cykl długości p będzie oznaczany przez C_p.

Do klasycznych problemów teorii grafów należy poszukiwanie warunków wystarczających na istnienie w grafach *cykli hamiltonowskich*, czyli cykli przechodzących przez wszystkie wierzchołki w grafie (graf, który zawiera taki cykl, nazywamy *hamiltonowskim*). Od lat 50-tych XX wieku znaleziono wiele różnych warunków wystarczających na hamiltonowskość grafu - najważniejsze z nich zostały zebrane w pracy Goulda [35]. Kilka z tych wyników zaprezentujemy tutaj (po więcej szczegółów odsyłamy czytelnika do rozdziału 4.2).

Twierdzenie Diraca z 1952 roku było pierwszym rezultatem w tej dziedzinie.

Twierdzenie 1.1 (Dirac [29]). *Niech G będzie grafem rzędu $n \geq 3$, w którym $\delta(G) \geq n/2$. Wówczas G zawiera cykl hamiltonowski.*

Osiem lat pózniej powyższy rezultat został w istotny sposób wzmocniony przez Orego, który udowodnił następujące twierdzenie.

Twierdzenie 1.2 (Ore [50]). *Niech G będzie grafem rzędu $n \geq 3$, w którym $d_G(x) + d_G(y) \geq n$ dla każdej pary wierzchołków niepołączonych x oraz y. Wtedy G jest grafem hamiltonowskim.*

Następny rezultat, który moglibyśmy nazwać warunkiem wystarczającym na rozmiar grafu dla hamiltonowskości, wynika natychmiast z twierdzenia 1.2.

Twierdzenie 1.3 (Ore [51]). *Niech G będzie grafem rzędu $n \geq 3$ oraz niech $||G|| > \binom{n-1}{2} + 1$. Wówczas G jest hamiltonowski.*

Erdős pokazał w 1962 roku, że warunek na rozmiar grafu może zostać osłabiony, jeśli dodamy dodatkowe ograniczenie na minimalny stopień grafu.

Twierdzenie 1.4 (Erdős [31]). *Niech G będzie grafem o rzędzie $n \geq 3$ i minimalnym stopniu $\delta(G) \geq r$, gdzie $1 \leq r \leq n/2$. Wtedy G zawiera cykl hamiltonowski, jeśli tylko*

$$\|G\| > \max\left\{\binom{n-r}{2} + r^2, \binom{n - \lfloor\frac{n-1}{2}\rfloor}{2} + \left\lfloor\frac{n-1}{2}\right\rfloor^2\right\}.$$

Wszystkie warunki wystarczające zaprezentowane w czterech powyższych twierdzeniach są ostre. Znane są grafy ekstremalne (patrz rozdział 4.2), które pokazują, że nie da się osłabić żadnego z założeń w twierdzeniach 1.1 - 1.4.

Z uwagi na autorów zaprezentowanych twierdzeń, warunki wystarczające, które podają ograniczenia na minimalny stopień grafu (odp. na sumę stopni par wierzchołków niepołączonych), będziemy nazywali *warunkami typu Diraca* (odp. *typu Orego*), natomiast warunki podające ograniczenia na rozmiar grafu jednocześnie z minimalnym stopniem grafu będziemy nazywali *warunkami typu Erdősa*.

Naturalne jest pytanie: jakie są warunki wystarczające (analogiczne do zaprezentowanych) na istnienie cykli innych długości w grafie?

Na przykład, odwołując się do twierdzenia 1.3, moglibyśmy sformułować następujący problem.

Problem 1. Dla dowolnej pary liczb naturalnych n oraz $k \leq n-3$ znaleźć minimalną liczbę naturalną $f(n,k)$ taką, że każdy graf o rzędzie równym n i rozmiarze większym niż $f(n,k)$ zawiera cykl długości $n-k$.

Oczywiście dla $k = 0$ jest to problem hamiltonowskości grafu rozwiązany przez Orego.

W roku 1972 Woodall podał rozwiązanie problemu 1 dla $n \geq 2k+3$ i co więcej udowodnił, że każdy graf rzędu n mający co najmniej $f(n,k)+1$ krawędzi zawiera cykle *wszystkich* długości od 3 do $n-k$ włącznie.

Twierdzenie 1.5 (Woodall [60]). *Niech G będzie grafem takim, że $|G| = n \geq 2k+3$ oraz*

$$||G|| > f(n,k) = \binom{n-k-1}{2} + \binom{k+2}{2}.$$

Wówczas G zawiera cykl długości p dla każdego $3 \leq p \leq n-k$.

Warunek na rozmiar podany przez Woodalla jest ostry. Graf składajacy się z dwóch klik, odpowiednio na $n-k-1$ oraz na $k+2$ wierzchołkach, mających dokładnie jeden wierzchołek wspólny, jest grafem ekstremalnym dla tego problemu. W oczywisty sposób nie zawiera on cyklu długości $n-k$ i ma dokladnie $f(n,k)$ krawędzi.

Założenie $n \geq 2k+3$ występujące w twierdzeniu 1.5 pokazuje, że Woodall znalazł uogólnienie twierdzenia 1.3 tylko dla długich cykli (dokładniej rzecz biorąc dla cykli C_p, gdzie $p \geq \frac{n+3}{2}$). To sugeruje, że przypadek krótkich cykli jest bardziej skomplikowany.

W istocie - jak dotąd problem jest rozwiązany tylko dla krótkich cykli długości nieparzystych. W roku 1907 Mantel [45] pokazał, że każdy graf na n wierzchołkach, mający więcej niż $\frac{n^2}{4}$ krawędzi, musi zawierać trójkąt. Bondy polepszył to twierdzenie w roku 1971.

Twierdzenie 1.6 (Bondy [22]). *Niech G będzie grafem na n wierzchołkach takim, że $||G|| > \frac{n^2}{4}$. Wtedy $C_p \subset G$ dla wszystkich $p \leq \frac{n+2}{2}$.*

Gdy n jest parzyste, wówczas graf dwudzielny pełny, który ma w każdym zbiorze bipodziału $n/2$ wierzchołków, jest grafem ekstremalnym pokazującym, że ograniczenie na rozmiar w twierdzeniu Bondy'ego jest najlepsze możliwe w przypadku nieparzystych $p \leq \frac{n+2}{2}$. W rzeczywistości kompletnie nie jest znana minimalna liczba krawędzi gwarantująca istnienie cyklu długości $p < \frac{n+3}{2}$, gdy p jest liczba parzystą, nawet w przypadku $p = 4$ (zobacz [23, 25]).

Ta szczególna różnica między cyklami długimi i krótkimi była główną inspiracją dla autora tej rozprawy, by przypatrzeć się bliżej długim cyklom. Specyficzna harmonia panująca w świecie długich cykli, odkryta przez Woodalla, skłania do rozważania podobnych problemów w innych rodzajach grafów.

Najpierw zajmujemy się problemem długich cykli w grafach dwudzielnych.

Niech zatem $G = (X, Y; E)$ oznacza graf dwudzielny z bipodziałem (X, Y) zbioru wierzchołków i zbiorem krawędzi E. Powiemy, że G jest *zrównoważony*, jeśli zbiory X oraz Y mają tyle samo elementów.

W rozdziale 5.1 prezentujemy różne znane warunki wystarczające na hamiltonowskość jak i bipancykliczność grafu (graf dwudzielny zrównoważony nazywamy *bipancyklicznym*, jeśli zawiera cykle wszystkich parzystych długości od 4 aż do rzędu grafu), będące odpowiednikami klasycznych twierdzeń przedstawionych w rozdziale 4.2. W rozdziale 5.2 podajemy nowe wyniki dotyczące warunku wystarczającego na rozmiar grafu dla istnienia długich cykli w grafach dwudzielnych, natomiast w rozdziale 5.3 rozważamy modyfikację wspomnianego warunku, czyli warunek typu Erdősa.

Wszystkie kryteria na hamiltonowskość przedstawione na początku tego rozdziału (twierdzenia 1.1 - 1.4) mają swoje odpowiedniki dla grafów dwudzielnych. W roku 1963 Moon i Moser udowodnili warunki wystarczające typu Diraca, Orego oraz Erdősa na istnienie cyklu hamiltonowskiego w grafie dwudzielnym zrównoważonym.

Twierdzenie 1.7 (Moon, Moser [48]). *Niech $G = (X, Y; E)$, gdzie $|X| = |Y| = n \geq 2$. Jeśli chociaż jeden z następujących warunków jest spełniony:*

(1) $\delta(G) \geq (n + 1)/2$,

(2) $d_G(x) + d_G(y) \geq n + 1$ *dla dowolnej pary niepołączonych wierzchołków $x \in X$ i $y \in Y$,*

(3) $||G|| > n(n - 1) + 1$,

(4) $\delta(G) \geq r$, *gdzie* $1 \leq r \leq n/2$, *oraz* $\|G\| > n(n - r) + r^2$,

to G jest grafem hamiltonowskim.

Przyjrzyjmy się trzeciemu warunkowi w tym twierdzeniu, czyli warunkowi wystarczającemu na rozmiar grafu dla hamiltonowskości. Naszym celem jest znalezienie uogólnienia tego warunku dla długich cykli, innymi słowy znalezienie odpowiednika twierdzenia Woodalla dla grafów dwudzielnych. Zatem interesuje nas rozwiązanie następującego problemu.

Problem 2. Dla grafu dwudzielnego $G = (X, Y; E)$ takiego, że $|X| = m \leq n = |Y|$, oraz dla naturalnego $k < \frac{m}{2}$, znaleźć minimalną liczbę krawędzi grafu G, $g(m, n, k)$, która gwarantuje istnienie cyklu długości $2m - 2k$ w G.

Zanim jednak przedstawimy rozwiązanie tego problemu, zdefiniujmy rodzinę $\mathcal{G}_{m,n,k}$ grafów dwudzielnych $G = (X, Y; E)$ na $m + n$ wierzchołkach, gdzie $|X| = m \leq n = |Y|$, $X = U \cup W$, $|W| = k + 1$, $d_G(u) = n$ dla każdego $u \in U$ oraz $d_G(w) = 1$ dla każdego $w \in W$. Zauważmy, że każdy graf z rodziny $\mathcal{G}_{m,n,k}$ ma rozmiar równy $n(m - k - 1) + k + 1$, a przy tym nie zawiera cyklu długości $2m - 2k$.

Problem 2 został częściowo rozwiązany, o czym świadczy poniższe twierdzenie (pełny dowód twierdzenia znajduje się w rozdziale 5.2).

Twierdzenie 1.8 (LA [4]). *Niech $G = (X, Y; E)$ będzie grafem dwudzielnym, w którym $|X| = m \leq n = |Y|$, gdzie $m \geq \frac{1}{2}k^2 + \frac{3}{2}k + 4$. Jeśli*

$$||G|| \geq g(m, n, k) = n(m - k - 1) + k + 1,$$

to albo G zawiera cykl długości $2m-2k$ albo $||G|| = g(m, n, k)$ i G jest izomorficzny z grafem z rodziny $\mathcal{G}_{m,n,k}$.
W obu przypadkach G zawiera cykle C_{2p} dla każdego $2 \leq p \leq m - k - 1$.

Zatem rozmiar grafu większy od $g(m, n, k)$ tak naprawdę gwarantuje istnienie cykli *wszystkich* parzystych długości od 4 do $2m - 2k$, podobnie jak to było w twierdzeniu 1.5.

Wydaje się jednak, że warunek na rząd grafu w twierdzeniu 1.8 można osłabić. Stawiamy następującą hipotezę.

Hipoteza 1.9. *Niech $G = (X, Y; E)$ będzie grafem dwudzielnym, w którym $|X| = m \leq n = |Y|$, gdzie $m \geq 2k + 2$. Jeśli $||G|| > n(m - k - 1) + k + 1$, to wówczas G zawiera cykle wszystkich parzystych długości od 4 do $2m - 2k$ włącznie.*

Zauważmy, że założenie $m \geq 2k + 2$ jest istotne, co wynika z poniższego przykładu.

Przykład 1.10. Niech $G_1 = (X, Y; E)$ będzie grafem dwudzielnym, w którym zbiory bipodziału X oraz Y są następującej postaci: $X = Q \cup R$, $Y = S \cup T$, gdzie $|Q| = |T| = m-k-1$, $|R| = k+1$, $|S| = n-m+k+1$, $m \leq n$. Ustalmy wierzchołek y_0 w T i niech $N_{G_1}(x) = S \cup \{y_0\}$ dla każdego $x \in Q$, natomiast $N_{G_1}(x) = T$ dla każdego $x \in R$. Wtedy $\|G_1\| > n(m - k - 1) + k + 1$ dla $k + 3 \leq m \leq 2k + 1$, ale G_1 nie zawiera żadnego cyklu długości większej niż $2m - 2k - 2$. Zatem założenie $m \geq 2k + 2$ w hipotezie 1.9 jest konieczne.

Ten przykład pokazuje również, że warunek wystarczający na rozmiar grafu podany w hipotezie 1.9, jeśli prawdziwy, dotyczy jedynie cykli długich. Podobnie jak to było w ogólnych grafach prostych, problem krótkich cykli w grafach dwudzielnych wydaje się być bardzo trudny (zobacz np. [12, 23, 39]).

Hipoteza 1.9 jest prawdziwa dla $k = 0$ oraz dla $k = 1$ i wydaje się, że jest prawdziwa dla dowolnego $k \leq \frac{m-2}{2}$ (zobacz rozdział 5.2).

Patrząc na jakikolwiek graf ekstremalny z rodziny $\mathcal{G}_{m,n,k}$ (zdefiniowanej tuż przed twierdzeniem 1.8), którego minimalny stopień jest równy 1, można zapytać: Jak zmieni się warunek wystarczający na rozmiar grafu dla istnienia cyklu długości $2m - 2k$, gdy dodatkowo założymy, że stopień minimalny grafu jest większy od 1? Próbą odpowiedzi na to pytanie jest następująca hipoteza (sformułowana tylko dla grafów dwudzielnych zrównoważonych).

Hipoteza 1.11. *Niech G będzie grafem dwudzielnym zrównoważonym rzędu $2n$ takim, że $\delta(G) \geq r \geq 1$, gdzie $n \geq 2k + 2r$, $k \in \mathbb{N}$. Jeśli*

$$\|G\| > n(n-k-r) + r(k+r),$$

to G zawiera cykl długości $2n-2k$.

Hipoteza 1.11 jest warunkiem typu Erdősa na istnienie długich cykli w grafach dwudzielnych zrównoważonych. Jest prawdziwa dla $k = 0$ - wówczas jest to dokładnie czwarty warunek na hamiltonowskość z twierdzenia 1.7. Najważniejszy wynik przedstawiony w rozdziale 5.3 to twierdzenie pokazujące, że hipoteza 1.11 jest prawdziwa także dla $k = 1$.

Zauważmy, że warunek na rozmiar występujący w hipotezie 1.11, jeśli prawdziwy, jest ostry, co pokazuje poniższy przykład.

Przykład 1.12. Niech G_2 będzie grafem dwudzielnym zrównoważonym, o klasach wierzchołków X i Y takich, że $|X| = |Y| = n$ oraz $X = Q \cup R$, $Y = S \cup T$, $|Q| = n-k-r$, $|R| = k+r$, $|S| = n-r$ i $|T| = r$. Co więcej, niech $N_{G_2}(x) = Y$ dla każdego $x \in Q$, natomiast $N_{G_2}(x) = T$ dla każdego $x \in R$. Wtedy $\delta(G_2) = r$, jeśli tylko $n-k-r \geq r$, zaś $\|G_2\| = n(n-k-r)+r(k+r)$. Jednakże, co łatwo zauważyć, G_2 nie zawiera cyklu długości $2n-2k$.

Na koniec wspomnijmy jeszcze, że w przypadku grafów prostych ogólnych (niekoniecznie dwudzielnych) warunek typu Erdősa na istnienie długiego cyklu został znaleziony przez Woodalla [60]. Uogólnił on twierdzenie 1.4 i podał pełną listę warunków na minimalny stopień i rozmiar grafu na n wierzchołkach gwarantujące zawieranie się cyklu długości $n-k$ dla dowolnego $0 \leq k \leq \frac{n-3}{2}$ (zobacz rozdział 4.2).

Rozdział 6 rozprawy jest poświęcony problemom dotyczącym długich cykli w grafach skierowanych. Okazuje się, że można zastosować twierdzenia 1.5 i 1.8 w celu uzyskania wyników dotyczących zawierania się wszystkich orientacji cyklu danej długości w digrafie. Zanim jednak przedstawimy konkretne rezultaty, wprowadzimy niezbędną terminologię i stosowaną w rozprawie notację.

Niech D będzie digrafem, V(D) zbiorem wierzchołków digrafu D, natomiast A(D) zbiorem łuków digrafu D. Wtedy $|D| = |V(D)|$ oznacza rząd digrafu D, zaś $||D|| = |A(D)|$ to jego rozmiar.

Niech x i y będą dwoma różnymi wierzchołkami w D. Niech $(x \to y)$ oznacza łuk w D skierowany od x do y. Powiemy, że x oraz y są *sąsiednie*, jeśli $(x \to y) \in A(D)$ lub $(y \to x) \in A(D)$. Mówimy, że sąsiednie wierzchołki x i y są połączone *łukiem symetrycznym*, jeśli zarówno $(x \to y) \in A(D)$ jak i $(y \to x) \in A(D)$; w przeciwnym wypadku mówimy, że są one połączone *łukiem antysymetrycznym*.

Dla wierzchołka $x \in V(D)$ definiujemy dwa zbiory: $N_D^+(x) = \{y \in V(D) : (x \to y) \in A(D)\}$ oraz $N_D^-(x) = \{y \in V(D) : (y \to x) \in A(D)\}$. Wówczas $d_D(x) = |N_D^+(x)| + |N_D^-(x)|$ to stopień wierzchołka x w D.

Digraf D jest *silnie spójny*, jeśli dla dowolnych dwóch wierzchołków $x, y \in V(D)$ istnieje w D ścieżka skierowana z x do y oraz ścieżka skierowana z y do x.

Ciąg $\epsilon = (\epsilon_1, ..., \epsilon_p)$, gdzie $\epsilon_i \in \{-1, 1\}$, $1 \leq i \leq p$, nazywamy *orientacją* cyklu $C = x_1...x_p x_1$ zawartego w D, jeśli $\epsilon_i = 1$ implikuje, że $(x_i \to x_{i+1}) \in A(D)$, natomiast $\epsilon_i = -1$ implikuje, że $(x_{i+1} \to x_i) \in A(D)$, dla każdego i (mod p). Wówczas C nazywamy *realizacją* ϵ w D. Każda realizacja w digrafie D orientacji $\epsilon = (\epsilon_1, ..., \epsilon_p)$, gdzie $\epsilon_i \epsilon_{i+1} = 1$ dla każdego i (mod p), jest nazywana *cyklem o silnej orientacji* długości p.
Przez C_p^* oznaczamy *cykl symetryczny* długości p, tj. cykl, którego wierzchołki są połączone łukami symetrycznymi, natomiast $C_p^{*'}$ oznacza C_p^* z usuniętym jednym łukiem i nazywamy go *cyklem prawie symetrycznym* długości p.

Wiele różnych warunków wystarczających na istnienie cykli o silnej orientacji zostało przedstawionych w przeglądowym artykule autorstwa Bermonda i Thomassena [19] (niektóre z prezentowanych tam wyników są cytowane w rozdziale 4.3). Wśród kryteriów na hamiltonowskość digrafów są między innymi dwa twierdzenia: rezultat Lewina z 1975 roku oraz twierdzenie z 1980 roku, którego autorami są Bermond, Germa, Heydemann i Sotteau.

Twierdzenie 1.13 (Lewin [44]). *Niech D będzie digrafem rzędu $n \geq 3$ takim, że $||D|| > (n-1)^2$. Wtedy D zawiera cykl hamiltonowski o silnej orientacji.*

Twierdzenie 1.14 (Bermond, Germa, Heydemann, Sotteau [18]). *Niech D będzie digrafem silnie spójnym rzędu $n \geq 5$, mającym co najmniej $(n-1)(n-2)+3$ łuki. Wówczas D zawiera cykl hamiltonowski o silnej orientacji.*

Oba te rezultaty zostały wzmocnione w roku 1982. Heydemann, Sotteau i Thomassen [42] odkryli, że założenia w twierdzeniach 1.13 i 1.14 implikują istnienie *dowolnej* orientacji cyklu hamiltonowskiego. Nawet więcej, Wojda [57] pokazał w 1986 roku, że te założenia dają zawieranie się każdej orientacji cyklu C_p w D dla każdego $3 \leq p \leq n$, jeśli tylko $n \geq 9$.

Wspomniane rezultaty zainspirowały autora rozprawy do rozważenia następującego problemu.

Problem 3. Dla dowolnej pary liczb naturalnych n oraz $k \leq n-3$, znaleźć minimalną liczbę naturalną $h(n,k)$ taką, że każdy digraf rzędu n, o rozmiarze większym niż $h(n,k)$, zawiera każdą orientację cyklu długości $n-k$.

Aby otrzymać rozwiązanie (dla długich cykli) tego problemu, dowodzimy twierdzenie 1.15 dotyczące cykli symetrycznych i prawie symetrycznych w digrafach, używając twierdzenia 1.5 (zobacz rozdział 6.1).

Najpierw jednak musimy zdefiniować pewne specjalne rodziny digrafów. Niech D_1 i D_2 będą dowolnymi rozłącznymi digrafami. Przez $D_1 \rightleftharpoons D_2$ oznaczamy rodzinę digrafów D takich, że $V(D) = V(D_1) \cup V(D_2)$ i dla każdych

$x, y \in V(D): \quad (x \to y) \in A(D)$ wtedy i tylko wtedy, gdy

(1) $x, y \in V(D_i)$ i $(x \to y) \in A(D_i)$ $(i = 1, 2)$ lub

(2) $x \in V(D_i), y \in V(D_j), i \neq j$ oraz $(y \to x) \notin A(D)$ (czyli wierzchołki z $V(D_1)$ są połączone z wierzchołkami z $V(D_2)$ łukami antysymetrycznymi).

Wśród digrafów z rodziny $D_1 \rightleftharpoons D_2$ wyróżniamy jeden konkretny: $D_1 \to D_2$, którego zbiór łuków składa się z łuków występujących w D_1, w D_2, oraz wszystkich łuków z D_1 do D_2. Niech $\mathcal{H}_{n,k}$ oznacza rodzinę $K^*_{n-k-1} \rightleftharpoons K^*_{k+1}$, gdzie K^*_m to digraf symetryczny pełny na m wierzchołkach. Na koniec, niech $D_1 * D_2$ oznacza digraf, w którym $V(D_1 * D_2) = V(D_1) \cup V(D_2)$, natomiast zbiór łuków składa się z łuków digrafów D_1 i D_2, a także z wszystkich łuków z D_1 do D_2 oraz z D_2 do D_1.

Jesteśmy gotowi, by podać twierdzenie o cyklach symetrycznych i prawie symetrycznych w digrafach.

Twierdzenie 1.15 (LA, Wojda [5]). *Niech D będzie digrafem na n wierzchołkach, gdzie $n \geq 7$, $n \geq \frac{5}{2}k + 6$, mającym co najmniej $(n-k-1)(n-1) + k(k+1)$ łuków. Wówczas:*

(1) D zawiera cykle symetryczne C^*_p dla każdego $3 \leq p \leq n-k-2$,

(2) D zawiera cykl prawie symetryczny $C^{*'}_{n-k-1}$; co więcej, jeśli $n \geq 3k+6$, to D zawiera cykl symetryczny C^*_{n-k-1},

(3) D zawiera cykl prawie symetryczny $C^{*'}_{n-k}$ pod warunkiem, że nie zachodzi żadna z następujących sytuacji:

(3a) D jest jednym z digrafów z rodziny $\mathcal{H}_{n,k}$,

(3b) $n = 3k+4$ i $D = T^{\rightleftharpoons}_{2k+3} * K^*_{k+1}$,

(3c) $n = 3k+2$ i $D = T^{\rightleftharpoons}_{2k+2} * K^*_k$,

gdzie $T^{\rightleftharpoons}_p$ to turniej rzędu p.

Konsekwencją powyższego twierdzenia jest następujący wniosek, który jest częściowym rozwiązaniem problemu 3.

Wniosek 1.16 (LA, Wojda [5]). *Niech D będzie digrafem takim, że $|D| = n$, $n \geq 7$, $n \geq \frac{5}{2}k + 6$, oraz $||D|| \geq h(n,k) = (n-k-1)(n-1) + k(k+1)$. Wówczas:*

(1) D zawiera każdą orientację cyklu długości $n-k$ za wyjątkiem silnej w przypadku, gdy $D = K^*_{n-k-1} \to K^*_{k+1}$ lub $D = K^*_{k+1} \to K^*_{n-k-1}$,

(2) D zawiera każdą orientację cyklu C_p dla każdego $3 \leq p \leq n-k-1$.

Warto nadmienić w tym miejscu, że powyższy wniosek jest częściowym uogólnieniem rezultatu Häggkvista i Thomassena z roku 1981 [19, Theorem 2.2.3]. Dla każdego $3 \leq p \leq n$ podali oni minimalny rozmiar digrafu D na n wierzchołkach, który gwarantuje istnienie silnej orientacji cyklu C_p w D. Wniosek 1.16 pokazuje, że ten sam rozmiar w rzeczywistości implikuje istnienie *dowolnej* orientacji cyklu C_p, jeśli tylko $p \geq \frac{3}{5}n + \frac{12}{5}$. Znalezienie podobnego uogólnienia dla cykli krótszych jest problemem otwartym.

W ostatnim rozdziale tej rozprawy rozważamy podobny problem do problemu 3, ale dla digrafów dwudzielnych.

Niech zatem $D = (X, Y; A)$ oznacza digraf dwudzielny ze zbiorami bipodziału X oraz Y, i zbiorem łuków A. Digraf D nazywamy *zrównoważonym*, gdy X i Y są tej samej liczności.

Niech $\mathcal{D}_{m,n,k} = \{\ D = (X, Y; A) :\ |X| = m \leq n = |Y|,\ X = U \cup W,\ U = \{x \in X : d_D(x) = 2n\},\ W = \{x \in X : d_D(x) = n\},\ |W| = k + 1,$ i każdy wierzchołek w W jest połączony ze wszystkimi wierzchołkami z Y łukami antysymetrycznymi$\}$.
Wyróżniamy specjalne digrafy $D^1_{m,n,k}$ i $D^2_{m,n,k}$ spośród digrafów z rodziny $\mathcal{D}_{m,n,k}$. Otóż w $D^1_{m,n,k}$ wszystkie łuki antysymetryczne pomiędzy wierzchołkami ze zbiorów W oraz Y są zorientowane z W do Y, podczas gdy w $D^2_{m,n,k}$ są one zorientowane z Y do W.

Problem 4. Dla digrafu dwudzielnego $D = (X, Y; E)$, gdzie $|X| = m \leq n = |Y|$, oraz naturalnego $k < \frac{m}{2}$, znaleźć minimalną liczbę łuków w D, $z(m, n, k)$, która gwarantuje istnienie każdej orientacji cyklu długości $2m - 2k$ w D.

Dla $m = n$ i $k = 0$ ten problem został rozwiązany w 1991 roku.

Twierdzenie 1.17 (Wojda, Woźniak [59]). *Niech $D = (X, Y; A)$ będzie digrafem dwudzielnym, w którym $|X| = |Y| = n \geq 3$ i $||D|| \geq 2n^2 - n$. Wtedy D zawiera każdą orientację cyklu hamiltonowskiego, chyba że $D = D^1_{n,n,0}$ lub $D = D^2_{n,n,0}$ (w obu przypadkach D nie zawiera silnej orientacji cyklu hamiltonowskiego).*

W ogólnym przypadku: $m \leq n$, $k \geq 0$, częściowo (dla $m \geq \frac{1}{2}k^2 + \frac{5}{2}k + 6$) rozwiązujemy postawiony problem (zobacz wniosek 1.19). Rozwiązanie jest wnioskiem z twierdzenia o cyklach symetrycznych i prawie symetrycznych w grafach dwudzielnych przedstawionego poniżej. Aby otrzymać ten rezultat, stosujemy twierdzenie 1.8 (szczegóły przedstawione są w rozdziale 6.2).

Twierdzenie 1.18 (LA [3]). *Niech $D = (X, Y; A)$ będzie digrafem dwudzielnym, w którym $|X| = m \leq n = |Y|$, gdzie $m \geq \frac{1}{2}k^2 + \frac{5}{2}k + 6$. Jeśli $||D|| \geq 2mn - n(k + 1)$, to:*

(1) D zawiera cykle symetryczne C^*_{2p} dla każdego $2 \leq p \leq m - k - 1$,

(2) D zawiera cykl prawie symetryczny $C^{*'}_{2m-2k}$ albo $D \in \mathcal{D}_{m,n,k}$.

Wniosek 1.19 (LA [3]). *Niech* $D = (X, Y; A)$ *będzie digrafem dwudzielnym takim, że* $|X| = m \leq n = |Y|$*, gdzie* $m \geq \frac{1}{2}k^2 + \frac{5}{2}k + 6$*. Przypuśćmy, że*

$$||D|| \geq z(m, n, k) = 2mn - n(k + 1).$$

Wówczas:

(1) D *zawiera każdą orientację cyklu długości* $2m - 2k$ *za wyjątkiem silnej w przypadku, gdy* $D = D^1_{m,n,k}$ *lub* $D = D^2_{m,n,k}$*,*

(2) D *zawiera każdą orientację cyklu* C_{2p} *dla każdego* $2 \leq p \leq m - k - 1$*.*

Sądzimy, że założenie $m \geq \frac{1}{2}k^2 + \frac{5}{2}k + 6$ w twierdzeniu 1.18 i wniosku 1.19 można osłabić, tzn. że m powinno być w zależności liniowej od k. Obecne ograniczenie kwadratowe na m jest prostą konsekwencją założeń twierdzenia 1.8, które zostało wykorzystane w dowodzie części (1) twierdzenia 1.18 (zobacz rozdział 6.2).
Istotnym założeniem na m w dowodzie części (2) twierdzenia 1.18 jest $m > 3k + 3$. Zatem warunek na m mógłby zostać osłabiony automatycznie, jeśli tylko potrafilibyśmy polepszyć rezultat z twierdzenia 1.8, na przykład udowadniając hipotezę 1.9.

Na koniec pokażemy, że ograniczenie na m nie może być mniejsze niż $2k+2$, konstruując specjalny digraf D_1.

Przykład 1.20. Niech $D_1 = (X, Y; E)$ będzie digrafem dwudzielnym, w którym zbiory bipodziału są postaci: $X = Q \cup R$, $Y = S \cup T$, gdzie $|Q| = |T| = m-k-1$, $|R| = k+1$, $|S| = n-m+k+1$, $m \leq n$. Niech $N^+_{D_1}(x) = Y$, $N^-_{D_1}(x) = S$ dla każdego $x \in Q$, i niech $N^+_{D_1}(x) = T$, $N^-_{D_1}(x) = Y$ dla każdego $x \in R$. Wtedy $\|D_1\| > z(m, n, k) = 2mn - n(k + 1)$ dla $k + 2 \leq m \leq 2k + 1$, natomiast D_1 nie zawiera silnej orientacji żadnego cyklu długości większej niż $2m - 2k - 2$.

Powyższy przykład pokazuje, że w digrafach dwudzielnych zagadnienie istnienia krótkich cykli wymaga zupełnie innego podejścia.

Chapter 2

French summary

Conditions suffisantes d'existence de longs cycles dans les graphes

Le but de cette thèse est de présenter des conditions suffisantes pour l'existence de longs cycles dans les graphes simples et les graphes orientés, c'est à dire des cycles passant par plus de la moitié des sommets d'un graphe donné. Plus précisément, nous voulons trouver la taille minimale d'un graphe donné G garantissant qu'un cycle de longueur prescrite est contenu dans G. On pourra considérer une modification de cette condition en ajoutant une borne sur le degré minimum de G. Nous étudions ce problème pour les graphes simples, en particulier les graphes bipartis et aussi les graphes orientés dans lesquels on considère toutes les orientations possibles d'un cycle de longueur donnée.

Tous les théorèmes présentés dans cette thèse sont accompagnés des noms des auteurs et de leur référence dans la littérature quand le résultat est publié. Les iniitiales LA indiquent que le résultat a pour auteur ou coauteur l'auteur de la présente thèse. Les théorèmes portant les initiales LA sont donnés ici avec leur démonstration complète.

Pour les définitions et la terminologie de base nous renvoyons le lecteur à [20, Chapter I]. Les notations utilisées dans cette thèse sont présentées dans la première section du Chapitre 4. En particulier, l'ordre d'un graphe G, c'est à dire son nombre de sommets, sera noté $|G|$, tandis que sa taille, ou nombre d'arêtes, sera noté $||G||$. Etant donné un sommet x de G, $N_G(x)$ sera l'ensemble des sommets adjacents à x dans G, $d_G(x)$ le degré de x dans G (i.e., $d_G(x) = |N_G(x)|$), et $\delta(G)$ le degré minimum pris sur tous les sommets de G. Un cycle de longueur p sera noté C_p.

Un des problèmes classiques en théorie des graphes est l'étude de conditions suffisantes pour qu'un graphe G contienne un *cycle hamiltonien*, c'est à dire un cycle qui passe par tous les sommets de G (on dit alors que G est *hamiltonien*). On présentera ici quelques unes des nombreuses conditions que l'on peut trouver dans la littérature (voir Section 4.2 pour plus de détails).

Le théorème de Dirac qui date de 1952 a été le premier résultat dans ce domaine.

Théorème 2.1 (Dirac [29]). *Soit G un graphe d'ordre $n \geq 3$, dans lequel $\delta(G) \geq n/2$. Alors G contient un cycle hamiltonien.*

Huit ans après ce résultat a été significativement amélioré par Ore qui a prouvé le théorème suivant.

Théorème 2.2 (Ore [50]). *Soit G un graphe d'ordre $n \geq 3$, dans lequel $d_G(x) + d_G(y) \geq n$ pour toute paire de sommets non adjacents x et y. Alors G est hamiltonien.*

Le résultat suivant, que l'on pourrait appeler condition d'hamiltonisme portant sur la taille, découle directement du théorème ci-dessus.

Théorème 2.3 (Ore [51]). *Soit G un graphe d'ordre $n \geq 3$, avec $||G|| > \binom{n-1}{2} + 1$. Alors G est hamiltonien.*

Erdős a montré en 1962 que la condition portant sur la taille peut être relaxée si l'on ajoute une borne sur le degré minimum du graphe

Théorème 2.4 (Erdős [31]). *Soit G un graphe d'ordre $n \geq 3$ et de degré minimum $\delta(G) \geq r$, où $1 \leq r \leq n/2$. Alors G contient un cycle hamiltonien pourvu que*

$$\|G\| > \max\left\{\binom{n-r}{2} + r^2, \binom{n - \lfloor \frac{n-1}{2} \rfloor}{2} + \left\lfloor \frac{n-1}{2} \right\rfloor^2\right\}.$$

Toutes les bornes dans ces quatre théorèmes sont les meilleures possibles. On connait des graphes extrêmaux qui montrent que les conditions ci-dessus ne peuvent pas être affaiblies.

En référence aux auteurs des théorèmes présentés ci-dessus, les conditions exprimant des contraintes sur le degré minimum (respectivement la somme des degrés de toute paire de sommets non adjacents) d'un graphe seront appeles *conditions de type Dirac* (resp. *conditions de type Ore*), tandis que celles qui combinent les bornes sur la taille d'un graphe avec des bornes sur son degré minimum seront appelées *conditions de type Erdős*.

Il est naturel de se poser la question de déterminer des conditions analogues à celles présentées ci-dessus et suffisantes pour assurer l'existence de cycles d'autres longueurs.

Par exemple, au vu du Théorème 2.3, il est naturel de généraliser ce problème comme suit : pour toute paire d'entiers non négatifs, n et $k \leq n-3$, trouver l'entier minimum $f(n,k)$ tel que tout graphe d'ordre n et de taille supérieure à $f(n,k)$ contienne un cycle de longueur $n-k$. Naturellement, pour $k=0$ on retrouve le problème hamiltonien.

En 1972, Woodall a déterminé $f(n,k)$ pour $n \geq 2k+3$ et, de plus, a prouvé que tout graphe d'ordre n et taille au moins $f(n,k)+1$ contient des cycles de *toutes* les longueurs jusqu'à $n-k$.

Théorème 2.5 (Woodall [60]). *Soit G un graphe vérifiant $|G| = n \geq 2k+3$ et*

$$||G|| > f(n,k) = \binom{n-k-1}{2} + \binom{k+2}{2}.$$

Alors G contient un cycle de longueur p pour tout p tel que $3 \leq p \leq n-k$.

Ce résultat est le meilleur possible. Le graphe consistant de deux cliques, l'une sur $n-k-1$ sommets et l'autre sur $k+2$ sommets, qui partagent un sommet, est un graphe extrêmal pour ce problème. Il ne contient aucun cycle de longueur $n-k$ et il a exactement $f(n,k)$ arêtes.

L'hypothèse $n \geq 2k+3$ montre que Woodall a trouvé une généralisation du Théorème 2.3 seulement pour les longs cycles (plus précisément, pour C_p avec $p \geq \frac{n+3}{2}$). Ceci suggère que le cas des cycles courts est plus difficile.

En fait, le problème est résolu seulement pour les cycles impairs. En 1907 Mantel [45] a montré que tout graphe à n sommets et plus de $\frac{n^2}{4}$ arêtes contient un triangle. Bondy a amélioré ce théorème en 1971.

Théorème 2.6 (Bondy [22]). *Soit G un graphe sur n sommets tel que $||G|| > \frac{n^2}{4}$. Alors $C_p \subset G$ pour tout $p \leq \frac{n+2}{2}$.*

Pour n pair, le graphe biparti complet avec $n/2$ sommets dans chaque classe montre que le résultat ci-dessus est le meilleur possible pour les valeurs impaires de $p \leq \frac{n+2}{2}$. En fait on ne sait pas complètement combien d'arêtes il faut dans un graphe pour assurer l'existence d'un cycle de longueur précisément égale à $p < \frac{n+3}{2}$, où p est pair, pas même pour $p = 4$ (voir [23, 25]).

Cette différence particulière entre cycles longs et courts a été l'inspiration principale qui a conduit l'auteur à regarder de près les longs cycles. L' harmonie spéciale découverte par Woodall dans le domaine des longs cycles invite à considérer des problèmes analogues pour d'autres types de graphes.

Nous étudions tout d'abord un problème de longs cycles dans les graphes bipartis.

Soit $G = (X, Y; E)$ un graphe biparti avec bipartition (X, Y) et ensemble d'arêtes E. Un tel graphe est appelé *équilibré* si X et Y ont même cardinal.

Dans la Section 5.1, nous présentons différentes conditions connues qui impliquent l'hamiltonisme ainsi que le bipancyclisme (un graphe biparti équilibré est dit *bipancyclique* s'il contient des cycles de toutes les longueurs paires entre 4 et son ordre). En Section 5.2, nous donnons des résultats concernant une condition suffisante d'existence de longs cycles dans les graphes bipartis portant sur la taille, tandis que dans la dernière section du Chapitre 5 nous discutons une modification de ce critère, qui est du type condition de Erdős.

Tous les critères d'hamiltonisme présentés au début de ce chapitre (Théorèmes 2.1 - 2.4) ont leurs analogues pour les graphes bipartis .

En 1963 Moon et Moser ont trouvé des conditions suffisantes de type Dirac, Ore, et Erdős pour l'existence d'un cycle hamiltonien dans un graphe biparti équilibré.

Théorème 2.7 (Moon, Moser [48]). *Soit $G = (X, Y; E)$ un graphe biparti équilibré avec $|X| = |Y| = n \geq 2$. Supposons que l'une des conditions suivantes est satisfaite:*

(1) $\delta(G) \geq (n + 1)/2$,

(2) $d_G(x) + d_G(y) \geq n + 1$ *pour toute paire de sommets non adjacent* $x \in X$ *et* $y \in Y$,

(3) $||G|| > n(n - 1) + 1$,

(4) $\delta(G) \geq r$, *où* $1 \leq r \leq n/2$, *et* $\|G\| > n(n - r) + r^2$.

Alors G est hamiltonien.

Examinons à nouveau le troisième critère qui porte sur la taille et suffit à assurer l'hamiltonisme. Notre but est de généraliser ce théorème aux longs cycles, c'est à dire d'obtenir un résultat analogue au théorème de Woodall pour les graphes bipartis. En d'autres termes, on veut résoudre le problème suivant : pour un graphe biparti $G = (X, Y; E)$ avec $|X| = m \leq n = |Y|$, et un entier $k < \frac{m}{2}$, trouver le nombre minimum d'arêtes de G, $g(m, n, k)$, qui garantit l'existence d'un cycle de longueur $2m - 2k$ dans G.

Avant d'introduire la solution, définissons la famille $\mathcal{G}_{m,n,k}$ de graphes bipartis $G = (X, Y; E)$ d'ordre $m + n$ tels que $|X| = m \leq n = |Y|$, $X = U \cup W$, $|W| = k + 1$, $d_G(u) = n$ pour tout $u \in U$, et $d_G(w) = 1$ pour tout $w \in W$. Notons que tout graphe de la famille $\mathcal{G}_{m,n,k}$ a pour taille $n(m - k - 1) + k + 1$ et ne contient aucun cycle de longueur $2m - 2k$.

Le problème ci-dessus est partiellement résolu, comme on peut le voir dans le théorème suivant (prouvé en Section 5.2).

Théorème 2.8 (LA [4]). *Soit $G = (X, Y; E)$ un graphe biparti avec $|X| = m \leq n = |Y|$, où $m \geq \frac{1}{2}k^2 + \frac{3}{2}k + 4$. Si*

$$||G|| \geq g(m, n, k) = n(m - k - 1) + k + 1,$$

alors soit G contient un cycle de longueur $2m - 2k$, ou sinon $||G|| = g(m, n, k)$ et G est isomorphe à un graphe de la famile $\mathcal{G}_{m,n,k}$.
Dans les deux cas, G contient C_{2p} pour tout $2 \leq p \leq m - k - 1$.

Ainsi la taille plus grande que $g(m, n, k)$ garantit, en fait, l'existence de cycles de *toutes* les longueurs paires jusqu'à $2m - 2k$, de même que dans le Théorème 2.5.

Cependant nous estimons que la condition sur l'ordre dans le Théorème 2.8 pourra être relaxée. Nous formulons la conjecture suivante.

Conjecture 2.9. *Soit $G = (X, Y; E)$ un graphe biparti avec $|X| = m \leq n = |Y|$, où $m \geq 2k + 2$. Si $||G|| > n(m - k - 1) + k + 1$, alors G contient des cycles de toutes les longueurs paires jusqu'à $2m - 2k$.*

Remarquons que l'hypothèse $m \geq 2k + 2$ de la conjecture est la plus faible possible, comme le montre l'exemple suivant.

Exemple 2.10. Soit $G_1 = (X, Y; E)$ un graphe biparti avec bipartition de la forme $X = Q \cup R$, $Y = S \cup T$, où$|Q| = |T| = m - k - 1$, $|R| = k + 1$, $|S| = n - m + k + 1$, $m \leq n$. Fixons un sommet y_0 dans T, et soit $N_{G_1}(x) = S \cup \{y_0\}$ pour tout $x \in Q$, et $N_{G_1}(x) = T$ pour tout $x \in R$. Alors $\|G_1\| > n(m-k-1)+k+1$ pour $k + 3 \leq m \leq 2k + 1$, cependant G_1 ne contient pas de cycle de longueur plus grande que $2m - 2k - 2$. d'où la nécessité de l'hypothèse $m \geq 2k + 2$.

Cet exemple montre aussi que, considérant des conditions suffisantes d'existence de cycles dans les graphes bipartis, nous avons à distinguer deux cas différents selon la longueur du cycle. A nouveau, comme dans le cas général des graphes simples, le problème des cycles courts dans les graphes bipartis semble aussi très difficile (voir par exemple [12, 23, 39]).

La conjecture 2.9 est vraie pour $k = 0$ et pour $k = 1$ et nous croyons qu'elle est vraie pour tout $k \leq \frac{m-2}{2}$ (voir Section 5.2).

Au vu des graphe extrêmaux de la famile $\mathcal{G}_{m,n,k}$ (définie juste avant le Théorème 2.8), dont le degré minimum vaut 1, on peut poser le problème suivant: donner la borne exacte pour la taille d'un graphe G d'ordre n assurante l'existence d'un cycle de longueur $2m-2k$, sachant que le degré minimum de G est plus grand que 1? Pour les graphes bipartis équilibrés (où $m = n$) nous proposons la conjecture suivante.

Conjecture 2.11. *Soit G un graphe biparti équilibré d'ordre $2n$ et de degré minimum $\delta(G) \geq r \geq 1$, où $n \geq 2k + 2r$ et $k \in \mathbb{N}$. Si*

$$\|G\| > n(n - k - r) + r(k + r),$$

alors G contient un cycle de longueur $2n - 2k$.

La conjecture 2.11 énonce une condition de type Erdős pour l'existence de longs cycles dans les graphes bipartis équilibrés et elle se montre vraie pour $k = 0$ (c'est alors exactement le quatrième critère d'hamiltonisme du Théorème 2.7). Le résultat principal de la Section 5.3 affirme que la Conjecture 2.11 est vraie pour $k = 1$ (voir aussi [1]).

Observons que la borne sur la taille dans la Conjecture 2.11 est étroite, comme on peut le voir dans l'Exemple 2.12 ci-dessous.

Exemple 2.12. Soit G_2 un graphe biparti équilibré avec bipartition X et Y, $|X| = |Y| = n$, où $X = Q \cup R$, $Y = S \cup T$, $|Q| = n - k - r$, $|R| = k + r$, $|S| = n - r$, and $|T| = r$. Supposons de plus que $N_{G_2}(x) = Y$ pour tout $x \in Q$, et $N_{G_2}(x) = T$ pour tout $x \in R$. Alors $\delta(G_2) = r$ si seulement $n - k - r \geq r$, et $\|G_2\| = n(n - k - r) + r(k + r)$. Cependant, comme on le voit facilement, G_2 ne contient pas de cycle de longueur $2n - 2k$.

Il faut mentionner ici que dans le cas général des graphes simples (pas nécessairement bipartis), la condition de type Erdős pour l'existence de longs cycles a été trouvée par Woodall [60]. Il a généralisé le Théorème 2.4 et donné une liste complète de condition de type Erdős pour qu'un graphe d'ordre n contienne un cycle de longueur $n - k$ pour tout $0 \leq k \leq \frac{n-3}{2}$ (voir Section 4.2).

Le Chapitre 6 de la thèse est consacré à l'étude des longs cycles dans les graphes orientés. Il s'avère que les Théorèmes 2.5 et 2.8 peuvent s'appliquer pour obtenir des résultats sur toutes les orientations d'un cycle de longueur donnée. Avant d'énoncer les résultats, nous introduisons les notations et la terminologie de base.

Soit D un graphe orienté, V(D) l'ensemble des sommets de D, A(D) l'ensemble des arcs de D. Alors $|D| = |V(D)|$ dénote l'ordre de D, tandis que $||D|| = |A(D)|$ est la taille de D.

Soient x et y deux sommets distincts de D. Soit $(x \rightarrow y)$ l'arc de D de x vers y. On dit que x et y sont *adjacents* si $(x \rightarrow y) \in A(D)$ ou $(y \rightarrow x) \in A(D)$. On dit que des sommets adjacents x et y sont joints par un *arc symétrique* si $(x \rightarrow y) \in A(D)$ et $(y \rightarrow x) \in A(D)$; on dit autrement qu'ils sont joints par un *arc antisymétrique*.

Pour un sommet $x \in V(D)$, on définit deux ensembles : $N^+_D(x) = \{y \in V(D) : (x \rightarrow y) \in A(D)\}$ et $N^-_D(x) = \{y \in V(D) : (y \rightarrow x) \in A(D)\}$. Alors $d_D(x) = |N^+_D(x)| + |N^-_D(x)|$ est le degré de x dans D.

Un graphe orienté D est *fort*, si pour toute paire de sommets x and y, D contient des chemins orientés de x vers y et de y vers x.

Une suite $\epsilon = (\epsilon_0, ..., \epsilon_{p-1})$, où $\epsilon_i \in \{-1, 1\}$, $0 \leq i \leq p - 1$, est appelée *l'orientation* d'un cycle $C = x_0...x_{p-1}x_0$ de D si $\epsilon_i = 1$ implique $(x_i \rightarrow x_{i+1}) \in A(D)$ et $\epsilon_i = -1$ implique $(x_{i+1} \rightarrow x_i) \in A(D)$ pour tout $i \pmod p$. Alors C est une *réalisation de* ϵ dans D. Toute réalisation dans un graphe orienté D de l'orientation $\epsilon = (\epsilon_0, ..., \epsilon_{p-1})$ est appelée *cycle fort* de longueur p si $\epsilon_i\epsilon_{i+1} = 1$ pour tout $i \pmod p$.

C_p^* dénote un *cycle symétrique* de longueur p, c'est à dire un cycle avec seulement des arcs symétriques, et $C_p^{*'}$ dénote C_p^* moins un arc, que nous appelons *cycle presque symétrique* de longueur p.

On trouve des conditions variées d'existence de cycles forts dans les graphe orientés dans le mémoire écrit par Bermond et Thomassen [19] (certaines sont mentionnées en Section 4.3). Parmi les critères d'hamiltonisme, il y a deux théorèmes: le résultat de Lewin de 1975, et le théorème de Bermond, Germa, Heydemann et Sotteau de 1980.

Théorème 2.13 (Lewin [44]). *Soit D un graphe orienté d'ordre $n \geq 3$ avec $||D|| > (n-1)^2$. Alors D contient un cycle fort hamiltonien .*

Théorème 2.14 (Bermond, Germa, Heydemann, Sotteau [18]). *Soit D un graphe orienté fort d'ordre $n \geq 5$ avec plus de $(n-1)(n-2)+2$ arcs. Alors D contient un cycle fort hamiltonien .*

Ces deux résultats ont été améliorés en 1982. Heydemann, Sotteau et Thomassen [42] ont découvert que les hypothèses des Théorèmes 2.13 et 2.14 impliquaient l'existence de toutes les orientations d'un cycle hamiltonien. Encore mieux, Wojda [57] a montré en 1986 que ces hypothèses fournissent chaque orientation de C_p dans D pour tout $3 \leq p \leq n$ si seulement $n \geq 9$. Ces résultats ont inspiré à l'auteur l'idée de considérer le problème suivant : pour toute paire d'entiers non negatifs n et $k \leq n-3$, trouver l'entier minimum $h(n,k)$, tel que tout graphe orienté d'ordre n et taille plus grande que $h(n,k)$ contienne toute orientation d'un cycle de longueur $n-k$.

Pour obtenir la solution de ce problème pour les longs cycles, nous prouvons le Théorème 2.15 sur les cycles symétriques et presque symétriques dans les graphes orientés utilisant le Théorème 2.5 (voir Section 6.1).

Mais nous devons d'abord définir des familles spéciales de graphe orientés. Soit D_1 and D_2 des graphes orientés sommets disjoints. $D_1 \rightleftharpoons D_2$ dénote la famille de graphes orientés D tels que $V(D) = V(D_1) \cup V(D_2)$ et pour tout $x, y \in V(D)$: $\quad (x \to y) \in A(D)$ si et seulemnet si

(1) $x, y \in V(D_i)$ et $(x \to y) \in A(D_i)$ $(i = 1, 2)$ ou

(2) $x \in V(D_i), y \in V(D_j), i \neq j$ et $(y \to x) \notin A(D)$ (les sommets de $V(D_1)$ sont joints avec les sommets de $V(D_2)$ par des arcs antisymétriques).

Parmi les graphes orientés de la famille $D_1 \rightleftharpoons D_2$ on en distingue un : $D_1 \to D_2$ dont l'ensemble des arcs est constitué des arcs de D_1 et D_2, et de tous les arcs de D_1 vers D_2. On notera $\mathcal{H}_{n,k}$ la famille $K^*_{n-k-1} \rightleftharpoons K^*_{k+1}$, où K^*_m dénote le graphe orienté complet symétrique avec m sommets.
Finalement, soit $D_1 * D_2$ le graphe orienté avec $V(D_1 * D_2) = V(D_1) \cup V(D_2)$ et dont les arcs sont les arcs de D_1 et D_2, et tous les arcs entre D_1 et D_2.

Nous sommes maintenant prêts à présenter le théorème sur les cycles symétriques et presque symétriques dans les graphes orientés.

Théorème 2.15 (LA, Wojda [5]). *Soit D un graphe orienté sur n sommets, où $n \geq 7$, $n \geq \frac{5}{2}k + 6$, et avec au moins $(n-k-1)(n-1) + k(k+1)$ arcs. Alors :*

(1) D contient tous les cycles symétriques C_p^* pour $3 \leq p \leq n-k-2$,

(2) D contient un cycle presque symétrique $C_{n-k-1}^{*'}$; de plus, si $n \geq 3k+6$ alors D contient un cycle symétrique C_{n-k-1}^*,

(3) D contient un cycle presque symétrique $C_{n-k}^{*'}$ sauf si

(3a) D est l'un des graphe orientés de $\mathcal{H}_{n,k}$,

(3b) $n = 3k+4$ et $D = T_{2k+3}^{\rightleftharpoons} * K_{k+1}^*$,

(3c) $n = 3k+2$ et $D = T_{2k+2}^{\rightleftharpoons} * K_k^*$,

où $T_p^{\rightleftharpoons}$ est un tournoi d'ordre p.

Comme conséquence du théorème ci-dessus nous obtenons le corollaire suivant.

Corollaire 2.16 (LA, Wojda [5]). *Soit D un graphe orienté. Supposons que $|D| = n$, $n \geq 7$, $n \geq \frac{5}{2}k + 6$ et*

$$||D|| \geq h(n,k) = (n-k-1)(n-1) + k(k+1).$$

Alors:

(1) D contient toute orientation d'un cycle de longueur $n-k$, sauf l'orientation forte dans le cas $D = K_{n-k-1}^* \to K_{k+1}^*$ ou $D = K_{k+1}^* \to K_{n-k-1}^*$,

(2) D contient toute orientation de C_p pour tout $3 \leq p \leq n-k-1$.

Il faut mentionner ici que le corollaire ci-dessus est une généralisation partielle d'un resultat d'Häggkvist et Thomassen de 1981 [19, Theorem 2.2.3]. Pour tout $3 \leq p \leq n$, ils ont trouvé la taille minimum d'un graphe orienté D à n sommets qui garantit l'existence d'une orientation forte d'un cycle C_p dans D. Le Corollaire 2.16 montre que la même taille implique, en fait, l'existence de chaque orientation d'un cycle C_p si seulement $p \geq \frac{3}{5}n + \frac{12}{5}$. La généralisation analogue pour les cycles plus courts semble difficile.

Dans la dernière section de la thèse on considère un problème similaire pour les graphes bipartis orientés.

Soit $D = (X, Y; A)$ un graphe biparti orienté avec bipartition X et Y, et ensemble d'arcs A. Un tel graphe orienté est dit *équilibré* si X et Y ont même cardinal.

Soit $\mathcal{D}_{m,n,k}$ la classe des graphes bipartis orientés $D = (X, Y; A)$, tels que $|X| = m \leq n = |Y|$, $X = U \cup W$, $U = \{x \in X : d_D(x) = 2n\}$, $W = \{x \in X :$

$d_D(x) = n\}$, $|W| = k+1$, et chaque sommet de W est joint à tous les sommets de Y par des arcs antisymétriques. On distingue les graphes orientés speciaux $D^1_{m,n,k}$ et $D^2_{m,n,k}$ de la famille $\mathcal{D}_{m,n,k}$: dans $D^1_{m,n,k}$ tous les arcs antisymétriques entre les sommets des ensembles W et Y sont orientés de W vers Y, alors que dans $D^2_{m,n,k}$ ils sont orientés de Y vers W.

Considérons le problème : pour un graphe biparti orienté $D = (X, Y; A)$ avec $|X| = m \leq n = |Y|$, et un entier $k < \frac{m}{2}$, trouver le nombre minimum d'arcs de D, $z(m, n, k)$, qui garantit l'existence de chaque orientation d'un cycle de longueur $2m - 2k$ dans D.

Pour $m = n$ et $k = 0$ ce problème a été résolu en 1991.

Théorème 2.17 (Wojda, Woźniak [59]). *Soit $D = (X, Y; A)$ un graphe biparti orienté, avec $|X| = |Y| = n \geq 3$, et $||D|| \geq 2n^2 - n$. Alors D contient chaque orientation d'un cycle hamiltonien à moins que $D = D^1_{n,n,0}$ ou $D = D^2_{n,n,0}$ (dans les deux cas D ne contient aucun cycle hamiltonien fort).*

Dans le cas général : $m \leq n$, $k \geq 0$, nous résolvons partiellement (pour $m \geq \frac{1}{2}k^2 + \frac{5}{2}k + 6$) le problème (voir Corollaire 2.19). La solution est une conséquence du théorème sur les cycles symétriques et presque symétriques dans les graphes bipartis. pour obtenir ce résultat nous appliquons le Théorème 2.8 (pour plus de details, voir Section 6.2).

Théorème 2.18 (LA [3]). *Soit $D = (X, Y; A)$ un graphe biparti orienté avec $|X| = m \leq n = |Y|$, où $m \geq \frac{1}{2}k^2 + \frac{5}{2}k + 6$. Si $||D|| \geq 2mn - n(k+1)$, alors:*

(1) D contient tous les cycles symétriques C^*_{2p} pour $2 \leq p \leq m - k - 1$,

(2) D contient un cycle presque symétrique $C^{*'}_{2m-2k}$ sauf si $D \in \mathcal{D}_{m,n,k}$.

Corollaire 2.19 (LA [3]). *Soit $D = (X, Y; A)$ un graphe biparti orienté avec $|X| = m \leq n = |Y|$, où $m \geq \frac{1}{2}k^2 + \frac{5}{2}k + 6$. Supposons que*

$$||D|| \geq z(m, n, k) = 2mn - n(k+1).$$

Alors :

(1) D contient chaque orientation d'un cycle de longueur $2m - 2k$, sauf l'orientation forte dans le cas $D = D^1_{m,n,k}$ ou $D = D^2_{m,n,k}$,

(2) D contient chaque orientation de C_{2p} pour tout $2 \leq p \leq m - k - 1$.

Nous pensons que l'hypothèse $m \geq \frac{1}{2}k^2 + \frac{5}{2}k + 6$ dans le Théorème 2.18 et le Corollaire 2.19 peut être relaxée; c'est à dire que m peut être supposé lineaire par rapport à k. La borne quadratique actuelle sur m est simplement une conséquence de l'hypothèse du Théorème 2.8, que nous utilisons dans la preuve de la partie (1) du Théorème 2.18 (voir Section 6.2).
L'hypothèse essentielle sur m dans la preuve de la partie (2) du Théorème 2.18 est $m > 3k + 3$. Ainsi la condition sur m serait automatiquement affaiblie si

seulement on pouvait améliorer le Théorème 2.8, par exemple en prouvant la Conjecture 2.9.

Finalement, nous montrons que la borne sur m ne peut pas être diminuée en dessous de $2k+2$ en construisant un graphe orienté special D_1 comme suit.

Exemple 2.20. Soit $D_1 = (X, Y; E)$ un graphe biparti orienté, avec bipartition de la forme $X = Q \cup R$, $Y = S \cup T$, où $|Q| = |T| = m-k-1$, $|R| = k+1$, $|S| = n-m+k+1$, $m \leq n$. Soit $N^+_{D_1}(x) = Y$, $N^-_{D_1}(x) = S$ pour tout $x \in Q$, et $N^+_{D_1}(x) = T$, $N^-_{D_1}(x) = Y$ pour tout $x \in R$. Alors $\|D_1\| > z(m,n,k) = 2mn - n(k+1)$ pour $k+2 \leq m \leq 2k+1$, cependant D_1 ne contient pas d'orientation forte d'un cycle de longueur plus grande que $2m-2k-2$.

Cet exemple montre, à nouveau, que les cycles courts demandent une approche différente.

Chapter 3

Introduction

The aim of this thesis is to present sufficient conditions for existence of long cycles in simple graphs and in directed graphs, that is, cycles which pass through more than half of the vertices in a given graph. Namely, we want to find the minimal size of a given graph G guaranteeing that a cycle of prescribed length is contained in G. Optionally we consider a modification of this condition by adding a bound on the minimal degree of G. We investigate this problem for simple graphs, particularly bipartite, and also for digraphs, where all possible orientations of a cycle of given length are considered.

All the theorems presented in this thesis are accompanied by the names of the authors and the reference to literature where the result is published. Initials LA indicate that the result was authored or co-authored by the author of the present thesis. Such theorems are given here with their complete proofs.

For the standard terminology we refer the reader to [20, Chapter I]. Basic defintions and notation used in this thesis are presented in Section 4.1.

One of the classical problems of graph theory is the study of sufficient conditions for a graph to contain a hamiltonian cycle. Many different conditions have been found. Let us mention some of them here.

Dirac's theorem [29] from 1952 was the first result in this field. He found a condition on the minimal degree of a graph G which guarantees that G is hamiltonian. In 1960, Ore [50] gave a lower bound on the degree sum of a pair of non-adjacent vertices in G ensuring hamiltonicity of G. This result has a simple consequence, proved by Ore [51], that every graph of order n and size greater than $\binom{n-1}{2} + 1$ has to contain a hamiltonian cycle. Of course, the bound on size is best possible in view of the graph which consists of a clique on $n - 1$ vertices and one pendent vertex. In 1962, Erdős [31] generalized this result adding an extra bound on the minimal degree of a graph.

All constraints in these results are best possible. There are known extremal graphs which show that none of the conditions can be weaken.

There is a natural question now: what are sufficient conditions analogous to the mentioned ones for existence of cycles of other lengths?

For example, in view of the size sufficient condition for hamiltonicity, it is natural to generalize this problem as follows: find the minimal size of a graph G on n vertices which guarantees that a cycle of length p ($3 \leq p \leq n$) is contained in G.

In 1972, Woodall [60] solved this problem in the case of $p \geq \frac{n+3}{2}$. He proved that every graph on n vertices which has more edges than the graph consisting of two cliques, one on $p-1$ vertices and the other one on $n-p+2$ vertices, which share one vertex, has to contain a cycle of length p. Thus his result is best possible.

The assumption $p \geq \frac{n+3}{2}$ shows that Woodall found a generalization of the hamiltonian size condition only for long cycles. This suggests that the case of short cycles is more difficult.

Indeed, the problem is solved only for odd cycles. In 1907, Mantel [45] showed that every graph G on n vertices and more than $\frac{n^2}{4}$ edges has to contain a triangle. Bondy [22] improved this theorem in 1971. He showed that the size greater than $\frac{n^2}{4}$ guarantees, in fact, existence of a cycle of length p in G for all $p \leq \frac{n+2}{2}$. For even n, the complete bipartite graph with $n/2$ vertices in each vertex class shows that Bondy's result is best possible for odd values of $p \leq \frac{n+2}{2}$. It is not completely known how many edges are needed in a graph to ensure the existence of a cycle of length precisely $p < \frac{n+3}{2}$, when p is even, not even for $p = 4$ (see [23, 25]).

This particular difference between long and short cycles was the main inspiration for the author to look closely at long cycles. The special harmony in the long cycles area discovered by Woodall in simple graphs invites to consider analogous problems in special families of graphs like bipartite graphs and in directed graphs (both general and bipartite).

Again, the problem of finding the minimal size of a bipartite graph G guaranteeing the existence of a small even cycle in G is very hard and still unsolved, even for C_4 and C_6 (see [12, 23, 39]). However, the analogous problem for long even cycles is not so hopeless.

Moon and Moser [48] found in 1963 the size sufficient condition for hamiltonicity of balanced bipartite graphs. In this thesis, we present a generalization of their theorem to long even cycles in general bipartite graphs. In other words, our result is an analogue of Woodall's theorem for bipartite graphs. We also consider a modification of this criterion by adding a bound on the minimal degree of a given graph, that is Erdős-type condition for long even cycles in bipartite graphs.

In the thesis, we investigate also analogous problems for long cycles in directed graphs. We are interested in finding the minimal size of a digraph (in particular bipartite) which guarantees existence of a long cycle of prescribed length in this digraph.

Many different sufficient conditions for existence of strong cycles in digraphs are presented in the survey written by Bermond and Thomassen [19]. Among criteria for hamiltonicity, there are two theorems. In 1975, Lewin [44] proved that every digraph of order n and size greater than $(n-1)^2$ has to contain a strong orientation of a hamiltonian cycle. Bermond, Germa, Heydemann and Sotteau [18] in 1980 found an analogous condition for strong digraphs. Both these results were strengthened several years later. Heydemann, Sotteau and Thomassen [42], and independently Wojda [57], discovered that the assumptions of the results by Lewin, and Bermond at al., in fact, imply existence of *every* orientation of a hamiltonian cycle. Their results inspired the author of this thesis to consider the general problem: what is the minimum size of a digraph of order n guaranteeing that every orientation of a long cycle of given length is contained in this digraph.

It turns out that Woodall's theorem and its bipartite counterpart (proved in the thesis) can be applied in order to get the solution of the above problem for both general and bipartite digraphs. First, we prove theorems about symmetric and almost symmetric cycles. The corollaries of them are theorems about all orientations of cycles in digraphs.

Outline of the thesis:

In **Chapter 4**, we present basic definitions and results existing in literature, connected with the topic of this thesis.

- In **Section 4.1**, we present the basic notions and notations used in the thesis.

- In **Section 4.2**, we give a list of chosen sufficient conditions for hamiltonicity as well as for pancyclicity. We introduce the notion of the Bondy-Chvátal closure, too. We also present two theorems by Woodall (Theorems 4.2.9 and 4.2.11) about long cycles in simple graphs, which were the inspiration for the author of this thesis.

- In **Section 4.3**, we quote some of various sufficient conditions for digraphs to be hamiltonian (or pancyclic), which are analogues of the theorems presented in the previous section. We give also the results mentioned above, due to Heydemann, Sotteau and Thomassen, and Wojda, about all orientations of hamiltonian cycles in digraphs. Finally, we study a history of the notion of an orientation of a cycle.

We investigate in **Chapter 5** a problem of long cycles in bipartite graphs.

- In **Section 5.1**, we present various known sufficent conditions for hamiltonicity of bipartite graphs, and bipancyclicity. They are the analogues of the results mentioned in Section 4.2. In particular, we introduce the notion of the biclosure, a very important tool used in the proof of the main result of Section 5.3.

- In **Section 5.2**, we give a size sufficient condition for existence of long cycles in bipartite graphs (Theorem 5.2.1). The result is presented with its proof. We discuss also how this result could be improved.

- In **Section 5.3**, we consider a modification of the criterion presented in the previous section, so called Erdős-type condition for long cycles in balanced bipartite graphs. We formulate a conjecture and prove it in a special case (Theorem 5.3.3).

Chapter 6 is devoted to the study of long cycles in directed graphs.

- In **Section 6.1**, we give a generalization of Wojda's result presented in Section 4.3. First, we prove the theorem about symmetric and almost symmetric cycles in digraphs (Theorem 6.1.1). Next we get the corollary about all orientations of long cycles in digraphs (Corollary 6.1.6). Finally, we compare the last result with the theorem by Häggkvist and Thomassen about strong orientations of cycles in digraphs.

- In **Section 6.2**, we deal with cycles in bipartite digraphs. We prove (applying Theorem 5.2.1) theorems about symmetric and almost symmetric cycles, and also about all orientations of cycles in bipartite digraphs. They are the bipartite counterparts of the results proved in the previous section.

Chapter 4

Preliminaries

4.1 Basic definitions and notation

In this section, we present the notions and notations used in the thesis. For the standard graph theoretic terminology we refer the reader to [20, Chapter I].

We consider first finite simple graphs without loops and multiple edges. For a graph G, we denote its vertex set by $V(G)$, and its edge set by $E(G)$. The *order* of G, which is the number of its vertices, is denoted by $|G|$, while the *size* of G, which is the number of its edges, will be denoted by $||G||$.

An edge $xy \in E(G)$ is said to *join* the vertices x and y of G. Then we say that x and y are *neighbours* or that they are *adjacent*; the vertices x and y are *incident* with the edge xy. A subset $U \subset V(G)$ consists of *independent vertices* if and only if no two elements of U are adjacent in G.
A graph on n vertices, of which all are adjacent, is called a *complete graph* of order n and is denoted by K_n. A *clique* is a complete subgraph of a given graph.

Two graphs are *isomorphic* if there is a correspondence between their vertex sets that preserves adjacency. Thus $G = (V, E)$ is isomorphic to $G' = (V', E')$ if there is a bijection $\phi : V \to V'$ such that $xy \in E$ if and only if $\phi(x)\phi(y) \in E'$. Then we write $G = G'$.

When we say that G *contains* a graph H, we mean that H is a subgraph of G.

For $U \subset V(G)$ and $x \in V(G)$, we denote by $e_G(x, U)$ the number of edges of G between the vertex x and vertices of the set U, by $N_G(U)$ the set of neigbours in G of all the vertices of U, and by $d_G(x)$ the degree of x in G (i.e., $d_G(x) = e_G(x, V(G)) = |N_G(x)|$). A vertex x of G is said to be *isolated* (resp. *pendent*) if $d_G(x) = 0$ (resp. $d_G(x) = 1$). The minimal vertex degree in G will be denoted by $\delta(G)$.
We write $G(U)$ for the subgraph of G spanned by U, so that $G - U = G(V(G) \setminus U)$ is the subgraph of G obtained by deleting the vertices in U

and all edges incident with them. If x and y are non-adjacent vertices of G, then $G + xy$ is obtained from G by joining x and y.

Given distinct vertices x and y of G, an *xy-path* is a path in G with end-vertices x and y. P_p denotes a path on p vertices; it is a path of length $p-1$. A cycle of length p, that is of order p, will be denoted by C_p. We call C_3 a *triangle*. A cycle is called *even* (resp. *odd*) when its length is even (resp. odd). A cycle (resp. path) passing through all the vertices of a graph G is said to be a *hamiltonian cycle* (resp. *hamiltonian path*) of G. A graph containing a hamiltonian cycle is called *hamiltonian*. We say that G is *pancyclic* if it contains cycles of all lengths from 3 up to the order of G.

We will denote by $G = (X, Y; E)$ a bipartite graph with vertex bipartition (X, Y), and the set of edges E. The sets X and Y are called *vertex classes*. G is called *balanced* when $|X| = |Y|$. A balanced bipartite graph is *bipancyclic* if it contains cycles of all even lengths from 4 up to its order.
We will denote by $K_{m,n}$ the *complete bipartite graph* with bipartition (X, Y) such that $|X| = m \leq n = |Y|$, that is the bipartite graph with all possible edges between vertices of the vertex classes X and Y.

Let us now turn to digraphs, again without loops and multiple arcs. For a digraph D, let V(D) be the set of vertices of D, and A(D) be the set of arcs of D. Then $|D| = |V(D)|$ denotes the order of D, while $||D|| = |A(D)|$ is the size of D.

Let x and y be two distinct vertices of D. By $(x \to y)$ we denote an arc of D from x to y. If the arc $(x \to y)$ is present, we say that x *dominates* y. We say that x and y are *adjacent* if $(x \to y) \in A(D)$ or $(y \to x) \in A(D)$. We say that adjacent vertices x and y are joined by a *symmetric arc* if both $(x \to y) \in A(D)$ and $(y \to x) \in A(D)$; otherwise we say that they are joined by an *antisymmetric arc*.
A *complete digraph* on n vertices, denoted by K_n^*, is the digraph in which all n vertices are adjacent and joined by symmetric arcs. We will denote by $\overline{D}$ the complement of D to the complete digraph with the vertex set $V(D)$.
An *oriented graph* is a digraph with no symmetric arc. A *tournament* on n vertices, denoted by $T_n^{\rightleftharpoons}$, is an oriented graph on n vertices in which any two vertices are adjacent, that is a digraph in which all n vertices are adjacent and they are joined by antisymmetric arcs only.
The *converse digraph* of D is the digraph obtained from D by reversing the directions of all arcs of D.

Similarly as it was in the case of simple graphs, for $U \subset V(D)$, $D(U)$ denotes the subdigraph of D induced by U, while $D - U$ stands for the subdigraph of D induced by $V(G) \setminus U$. For subsets U and W of $V(D)$, by $a_D(U \to W)$ we denote the number of arcs of D from U to W, and $a_D(U, W) = a_D(U \to W) + a_D(W \to U)$.

Given a vertex $x \in V(D)$, we define two sets: $N_D^+(x) = \{y \in V(D) : (x \to y) \in A(D)\}$ and $N_D^-(x) = \{y \in V(D) : (y \to x) \in A(D)\}$. Then the *outdegree* $d_D^+(x)$ of x in D is the cardinality of $N_D^+(x)$, the *indegree* $d_D^-(x)$ of x in D is the cardinality of $N_D^-(x)$, while the *degree* of x in D is $d_D(x) = d_D^+(x)+d_D^-(x)$ (i.e., $d_D(x) = a_D(x, V(D))$. A vertex x of D is called *source* (resp. *sink*) if $d_D^-(x) = 0$ (resp. $d_D^+(x) = 0$). The minimal outdegree (resp. minimal indegree) in D will be denoted by $\delta^+(D)$ (resp. $\delta^-(D)$), while $\delta(D)$ will denote the minimal degree of D.

A sequence $\epsilon = (\epsilon_1, ..., \epsilon_p)$, where $\epsilon_i \in \{-1, 1\}$, $1 \leq i \leq p$, is called the *orientation* of a cycle $C = x_1...x_px_1$ of D if $\epsilon_i = 1$ implies $(x_i \to x_{i+1}) \in A(D)$ and $\epsilon_i = -1$ implies $(x_{i+1} \to x_i) \in A(D)$ for every i (mod p). Then C is called a *realization of* ϵ in D. Any realization in a digraph D of the orientation $\epsilon = (\epsilon_1, ..., \epsilon_p)$ is called a *strong cycle* of length p, denoted by $C_p^{\to}$, if $\epsilon_i\epsilon_{i+1} = 1$ for every i (mod p), and an *antidirected cycle* of length p if $\epsilon_i\epsilon_{i+1} = -1$ for every i (mod p). Of course, in the latter case p is even.
C_p^* denotes a *symmetric cycle* of length p, that is a cycle with symmetric arcs only, and $C_p^{*'}$ denotes C_p^* minus one arc, which we call an *almost symmetric cycle* of length p (see Figure 1).

We say that a digraph is *hamiltonian* if it contains a strong hamiltonian cycle. A digraph D is *pancyclic* if it contains strong cycles of all lengths from 3 up to the order of D.

We denote by P_p^* the *symmetric path* of order p, that is a path with symmetric arcs only.

A digraph D is *strong*, if for any two vertices x and y, D contains a directed path from x to y and a directed path from y to x.

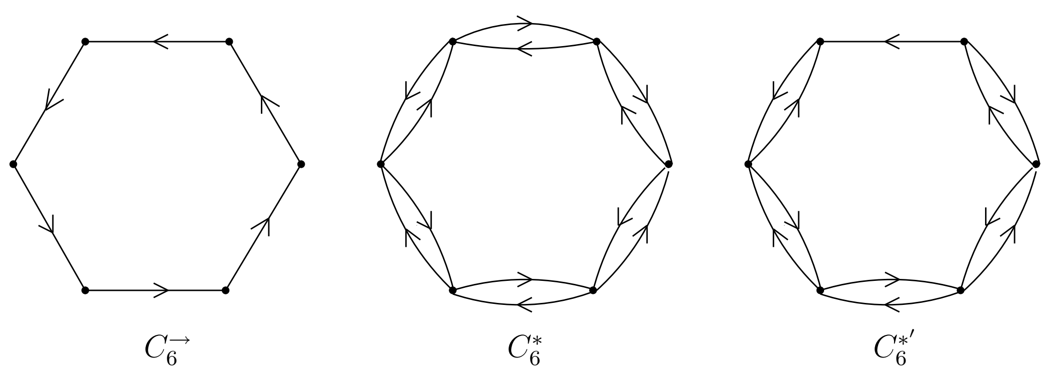


Figure 1

We will denote by $D = (X, Y; A)$ a bipartite digraph with vertex classes X and Y, and the set of arcs A. Again, such a digraph D is called *balanced* if X and Y have equal cardinalities. A balanced bipartite digraph is *bipancyclic* if it contains strong cycles of all even lengths from 4 up to its order.
We will denote by $K^*_{m,n}$ the *complete bipartite digraph* with bipartition (X, Y) such that $|X| = m \leq n = |Y|$, that is the bipartite digraph with all possible symmetric arcs between vertices of the vertex classes X and Y. A *bipartite tournament* with vertex bipartition (X, Y) such that $|X| = m \leq n = |Y|$, denoted by $T^{\rightleftharpoons}_{m,n}$, is an oriented graph in which any two vertices of different vertex classes X and Y are adjacent.
Finally, let $\overline{D^{bip}}$ denote the complement of $D = (X, Y; A)$ to the complete bipartite digraph $K^*_{|X|,|Y|}$.

4.2 Long cycles in simple graphs

Graph theorists have been searching for a long time for conditions ensuring that a graph is hamiltonian (see for example [35]). We shall present here some of many conditions existing in literature.

Most of the known results are descendants of Dirac's theorem published in 1952.

Theorem 4.2.1 (Dirac [29]). *Let G be a graph of order $n \geq 3$, in which $\delta(G) \geq n/2$. Then G contains a hamiltonian cycle.*

Eight years later this result was significantly strengthened by Ore who proved the following theorem.

Theorem 4.2.2 (Ore [50]). *Let G be a graph of order $n \geq 3$, in which $d_G(x) + d_G(y) \geq n$ for every pair of non-adjacent vertices x and y. Then G is hamiltonian.*

Next result, which we could call a size condition for hamiltonicity, follows immediately from the above theorem.

Theorem 4.2.3 (Ore [51]). *Let G be a graph of order $n \geq 3$ with $||G|| > \binom{n-1}{2} + 1$. Then G is hamiltonian.*

Erdős showed in 1962 that the condition on size can be lowered if we add a bound on the minimal degree of a graph.

Theorem 4.2.4 (Erdős [31]). *Let G be a graph of order $n \geq 3$ and minimal degree $\delta(G) \geq r$, where $1 \leq r \leq n/2$. Then G contains a hamiltonian cycle, provided*

$$\|G\| > \max\left\{\binom{n-r}{2} + r^2, \binom{n - \lfloor\frac{n-1}{2}\rfloor}{2} + \left\lfloor\frac{n-1}{2}\right\rfloor^2\right\}.$$

Remark 4.2.5. All bounds in these four theorems are best possible. Consider the graph G which consists of a complete graph on $n-r$ vertices, r of which are joined to each of r independent vertices. G has n vertices and $\binom{n-r}{2}+r^2$ edges, $\delta(G) \geq r$, and G is not hamiltonian. This example shows that the condition in Theorem 4.2.4 cannot be weakened. Take $r=1$ (resp. $r=\lfloor\frac{n-1}{2}\rfloor$) in this example to see that the conditions in Theorems 4.2.3 and 4.2.2 (resp. Theorem 4.2.1) cannot be improved either.

Referring to the authors of the theorems presented above, conditions that put constraints on minimal degree (resp. degree sums of pairs of non-adjacent vertices) of a graph will be called *Dirac-type* (resp. *Ore-type*) *conditions*, while those that combine bounds on the size of a graph with bounds on its minimal degree will be called *Erdős-type conditions*.

Bondy discovered in 1971 that many conditions that are sufficient for a graph to be hamiltonian are also sufficient for the graph to be pancyclic. For example, the conditions presented in Theorems 4.2.1 - 4.2.4, in fact, imply that either a graph G contains cycles of all lengths up to the order of G or else $G = K_{\frac{n}{2},\frac{n}{2}}$. It is a simple corollary of the following result.

Theorem 4.2.6 (Bondy [21]). *Let G be a hamitonian graph on n vertices such that $||G|| \geq \frac{n^2}{4}$. Then either G is pancyclic or n is even and G is the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$.*

We want to present here also another sufficient condition for hamiltonicity. Bondy and Chvátal [24] discovered that Ore, proving Theorem 4.2.2, actually proved more. Indeed, if x and y are non-adjacent vertices of a graph G of order n, satisfying the condition: $d_G(x)+d_G(y) \geq n$, and $G+xy$ has a hamiltonian cycle, then G, too, has one. This fact inspired Bondy and Chvátal to introduce the notion of the closure of a given graph. The *k-closure* $Cl_k(G)$ is obtained from G by recursively joining pairs of non-adjacent vertices whose degree sum is at least k, until no such pair remains. The k-closure is independent of the order of adjunction of the edges.

Let $\mathcal{P}$ be a property defined on all graphs of order n; let k be a nonnegative integer. Then $\mathcal{P}$ is said to be *k-stable* if, whenever $G+xy$ has the property $\mathcal{P}$ and $d_G(x)+d_G(y) \geq k$, then G itself has $\mathcal{P}$. The smallest integer k such that $\mathcal{P}$ is k-stable is called the *stability* of $\mathcal{P}$. The closure theory is based on the fact that if the property $\mathcal{P}$ is k-stable and if $Cl_k(G)$ satisfies $\mathcal{P}$, then G itself satisfies $\mathcal{P}$.

Bondy and Chvátal showed that the property of containing a hamiltonian cycle is n-stable.

Theorem 4.2.7 (Bondy and Chvátal [24]). *A graph G of order n is hamiltonian if and only if $Cl_n(G)$ is hamiltonian.*

There is no analogous result for pancyclicity. In fact, the stability of the property $\mathcal{P}$: "G is pancyclic" is unknown. For sure, $\mathcal{P}$ is not n-stable. If n is even, the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$ has n-closure complete and contains no odd cycles. However, Faudree et al. obtained the following result.

Theorem 4.2.8 (Faudree, Favaron, Flandrin and Li [33]). *Let G be a graph of order n such that $Cl_{n+1}(G) = K_n$. Then G is pancyclic.*

Now, let us look again at Theorem 4.2.3. It is natural to ask about a generalization of this result for cycles of other lengths. Therefore we consider the following problem.

Problem 1. For any pair of nonnegative integers n and $k \leq n-3$, find the minimum integer $f(n,k)$, such that every graph of order n and size greater than $f(n,k)$ contains a cycle of length $n-k$.

Of course, for $k = 0$ it is the hamiltonian problem (solved by Ore).

In 1972, Woodall found the number $f(n,k)$ for $n \geq 2k+3$ and, what's more, proved that every graph of order n and size at least $f(n,k)$ contains cycles of *all* lengths up to $n-k$. Thus Woodall generalized Theorem 4.2.3 and its modification due to Bondy (about pancyclicity).

Theorem 4.2.9 (Woodall [60]). *Let G be a graph with $|G| = n \geq 2k+3$, $k \in \mathbb{N}$, and*

$$||G|| \geq f(n,k) = \left(\begin{array}{c} n-k-1 \\ 2 \end{array} \right) + \left(\begin{array}{c} k+2 \\ 2 \end{array} \right) + 1.$$

Then G contains a cycle of length p for each p such that $3 \leq p \leq n-k$.

Remark 4.2.10. The above result is best possible. The graph consisting of two cliques, one on $n-k-1$ vertices and the other one on $k+2$ vertices, which share one vertex, is an extremal graph for this problem. It does not contain any cycle of length $n-k$ if $n-k \geq k+3$, and it has exactly $f(n,k)$ edges.

Woodall, proving Theorem 4.2.9, answered the question formulated by Erdős in 1969. Erdős [32] showed that for $n > n_0(k)$ every graph G of order n and size greater than $\binom{n-k-1}{2} + \binom{k+2}{2}$ contains a cycle of length $n-k$. He wanted to determine or estimate $n_0(k)$. Erdős' challenge was taken up by Woodall, who proved that $n_0(k) = 2k+2$.

Woodall generalized also Theorem 4.2.4. He found an Erdős-type condition for existence of long cycles in graphs. He gave a complete list of the conditions on minimal degree and size for a graph of order n to contain a cycle C_{n-k} for any $0 \leq k \leq \frac{n-3}{2}$.

For convenience, before presenting the theorem, we will define two functions: $f_1(n,m) = \binom{n-m}{2} + \binom{m+1}{2}$, and $f_2(n,m,l) = \binom{n-m-l}{2} + m(m+l)$.

Theorem 4.2.11 (Woodall [60]). *Let G be a graph with $|G| = n \geq 2k+3$, $k \in \mathbb{N}$, and $\delta(G) \geq r$. Then G contains a cycle of length p for each p such that $3 \leq p \leq n-k$, provided that G has at least the number of edges specified below.*
We distinguish five possible cases depending on the value of r:

(1) $r \leq k+1$; *then* $||G|| \geq f_1(n, k+1) + 1$,

(2) $k+1 \leq r < \frac{n-k}{2}$;
then $||G|| \geq \max\left\{f_1(n,r),\ f_2(n,r,k),\ f_2(n, \lfloor \frac{n-k-1}{2} \rfloor, r)\right\} + 1$,

(3) $\max\left\{k+1,\ \frac{n-k}{2}\right\} \leq r < \frac{n}{2}$; *then* $||G|| \geq f_1(n,r) + 1$,

(4) $r = \frac{n}{2}$; *then* $||G|| \geq \lfloor \frac{n^2}{4} \rfloor + 1$,

(5) $r > \frac{n}{2}$; *then* $||G|| \geq 0$ *(i.e., no bound needed).*

Remark 4.2.12. All presented bounds on size of G are best possible. Woodall [60] characterized extremal graphs for all five cases of Theorem 4.2.11. Of course, the cases (4) and (5) are simple consequences of Theorems 4.2.1 and 4.2.6.

Woodall's theorems presented above were the inspiration for the author of this thesis. In the next chapters, we will investigate similar problems to Problem 1 in bipartite graphs, and also in general and bipartite digraphs. We are interested in finding analogues of Theorems 4.2.9 and 4.2.11.

4.3 Orientations of cycles in digraphs

Most of the theorems about sufficient conditions for hamiltonicity and pancyclicity quoted in the previous section have the counterparts in the world of directed graphs. Let us present them now here.

The following two theorems are Dirac-type conditions for digraphs to be hamiltonian.

Theorem 4.3.1 (Ghouila-Houri [34]). *Let D be a strong digraph of order $n \geq 3$ with $\delta(D) \geq n$. Then D is hamiltonian.*

Theorem 4.3.2 (Nash-Williams [49]). *Let D be a digraph on n vertices ($n \geq 3$) such that $\delta^+(D) \geq n/2$ and $\delta^-(D) \geq n/2$. Then D contains a strong hamiltonian cycle.*

Next two results are Ore-type conditions for hamiltonicity of a digraph.

Theorem 4.3.3 (Woodall [60]). *A digraph D of order $n \geq 3$ is hamiltonian if, for any two vertices x and y, either x dominates y or $d^+_D(x) + d^-_D(y) \geq n$.*

Theorem 4.3.4 (Meyniel [46]). *A strong digraph D of order $n \geq 3$ is hamitonian if for any two non-adjacent vertices x and y we have $d_D(x) + d_D(y) \geq 2n - 1$.*

Remark 4.3.5. All four theorems above are best possible in the sense that they become false if the degree conditions are relaxed. This can be demonstrated by the complete bipartite digraphs with the property that the difference between the cardinalities of the vertex classes is one.
The requirement in Theorems 4.3.1 and 4.3.4 that D be strong is necessary as is demonstrated by a digraph consisting of two complete digraphs joined completely by antisymmetric arcs all in the same direction.

The following result generalizes Theorems 4.3.1 and 4.3.2. It is a Dirac-type condition for pancyclicity of a digraph.

Theorem 4.3.6 (Häggkvist and Thomassen [41]). *Let D be a strong digraph of order $n \geq 3$ such that $\delta(D) \geq n$. Then D is pancyclic or else n is even and D is isomorphic to $K^*_{\frac{n}{2},\frac{n}{2}}$.*

Thomassen proved an Ore-type condition for pancyclicity.

Theorem 4.3.7 (Thomassen [56]). *Let D be a strong digraph of order $n \geq 3$ such that for any two non-adjacent vertices x and y of D we have $d_D(x) + d_D(y) \geq 2n$. Then D is pancyclic or else n is even and $D = K^*_{\frac{n}{2},\frac{n}{2}}$.*

This result extends Woodall's theorem (Theorem 4.3.3). However, it does not include Meyniel's theorem (Theorem 4.3.4), and it becomes false if we replace the degree condition by Meyniel's condition. To see this consider the digraph $D_{n,l}$ with vertex set $\{x_1, x_2, \ldots, x_n\}$ and arc set $\{(x_i \to x_j) : i < j \text{ or } i = j + 1\} \setminus \{(x_i \to x_{i+l-1}) : 1 \leq i \leq n - l + 1\}$. Then $D_{n,l}$ satisfies Meyniel's condition and, moreover, it has only two pairs of non-adjacent vertices, $\{x_1, x_l\}$ and $\{x_{n-l+1}, x_n\}$, such that the inequality in Meyniel's theorem is an equality.
Benhocine [14] characterized all digraphs which satisfy Meyniel's condition and are not pancyclic.

For later purposes, we observe that the number of arcs of $D_{n,l}$ is $\frac{n(n-1)}{2}+l-2$.

In 1984, Amar et al. gave an Erdős-type condition for hamiltonicity of a digraph.

Theorem 4.3.8 (Amar, Fournier and Germa [8]). *Let D be a digraph of order $n \geq 3$ with $\delta^+(G) \geq r$, $\delta^-(G) \geq r$. If $||D|| > n(n-1) - (r+1)(n-r-1)$, then D is hamiltonian.*

Four years later Wojda [58] characterized all non-hamiltonian digraphs of order $n \geq 3$ with $\delta^+(G) \geq r$, $\delta^-(G) \geq r$, and precisely $n(n-1)-(r+1)(n-r-1)$ arcs.

Theorem 4.3.8 is, of course, a generalization (take $r = 0$) of the following result which is a size sufficient condition for hamiltonicity.

Theorem 4.3.9 (Lewin [44]). *Let D be a digraph of order $n \geq 3$ with $||D|| > (n-1)^2$. Then D contains a strong hamiltonian cycle.*

We can decrease the bound on size when we assume that D is strong.

Theorem 4.3.10 (Bermond, Germa, Heydemann, Sotteau [18]). *Let D be a strong digraph of order $n \geq 5$ with more than $(n-1)(n-2)+2$ arcs. Then D contains a strong hamiltonian cycle.*

Bondy [21] conjectured that every hamiltonian digraph of order n with at least $\frac{n^2}{2}$ arcs is pancyclic unless n is even and the digraph is isomorphic to $K^*_{\frac{n}{2},\frac{n}{2}}$. However, the digraph $D_{n,n-1}$ defined above is hamiltonian, but not pancyclic, and it has $\frac{1}{2}n(n+1)-3$ arcs. On the other hand we have the following result.

Theorem 4.3.11 (Häggkvist and Thomassen [41]). *A hamiltonian digraph D with n vertices and $\frac{1}{2}n(n+1)-1$ or more arcs is pancyclic.*

Combining this theorem with Theorems 4.3.9 and 4.3.10 we get the following corollary.

Corollary 4.3.12 (Häggkvist and Thomassen [41, 19]).

(1) A digraph with n vertices and $(n-1)^2+1$ or more arcs is pancyclic.

(2) A strong digraph with n vertices and $(n-1)(n-2)+3$ or more arcs is pancyclic.

Now, we want to investigate generalizations (to other orientations of cycles) of some of the presented results. First, let us discuss a size sufficient condition.

Theorems 4.3.9 and 4.3.10 were strengthened in 1982. Heydemann et al., improving a result of Benhocine and Wojda [15] and answering their conjecture, got the following result.

Theorem 4.3.13 (Heydemann, Sotteau and Thomassen [42]). *A digraph with n vertices, $n \geq 5$, and at least (n-1)(n-2)+3 arcs contains all orientations of a hamiltonian cycle except possibly the strong one .*

As we see, the assumptions in Theorems 4.3.9 and 4.3.10 imply, in fact, existence of *every* orientation of a hamiltonian cycle. Even more, Wojda showed in 1986 that these assumptions give every orientation of C_p in D for all $3 \leq p \leq n$.

Theorem 4.3.14 (Wojda [57]). *If D is a digraph with n vertices, $n \geq 9$, and at least (n-1)(n-2)+3 arcs, then D contains every orientation of a cycle of length at most n except the strong cycle of length n when D contains a sink or a source.*

This theorem has a simple consequence, which is a generalization of the first part of Corollary 4.3.12.

Corollary 4.3.15 (Wojda [57]). *Let D be a digraph of order $n \geq 9$ with $||D|| \geq (n-1)^2$. Then D contains every orientation of a cycle of length p for all $3 \leq p \leq n$ unless D is the digraph consisting of K^*_{n-1} plus a new vertex which dominates or is dominated by all the vertices of the K^*_{n-1}.*

In Chapter 6, we find generalizations of this result to long cycles both in general and bipartite digraphs.

Let us now turn to degree conditions. Grant [36] was first who considered Dirac-type condition for existence of an antidirected hamiltonian cycle. He conjectured that every digraph of order n and minimal indegree and outdegree at least $\frac{1}{2}n$ has an antidirected hamiltonian cycle. This conjecture was disproved by Cai [26]. His digraph is also a counterexample to the statement that "a digraph of order n, such that the sum of the degrees for any two non-adjacent vertices is at least $2n-1$, contains all orientations of a hamiltonian cycle, except possibly the strong one". It seems to be hard to find a generalization of Meyniel's condition, even replacing $2n-1$ by $2n$ (see [13, 17]). It remains an open problem to find a general degree sufficient condition for existence of every orientation of a cycle of given length (even for hamiltonian case).

Finally, let us mention that the notion of an orientation of a cycle is not so young and was defined first time for tournaments.

The fundamental early result on strong hamiltonian cycles in tournaments is that of Camion [27]: a tournament has a strong hamiltonian cycle if and only if the tournament is strong. In 1974, Rosenfeld [52] conjectured that there is an integer $N > 8$ such that every tournament of order $n \geq N$ contains every non-strong cycle of length n. His conjecture was a generalization of Grünbaum's conjecture [37] about antidirected hamiltonian cycles in tournaments, which was first proved by Thomassen[55]. Benhocine and Wojda [16] verified Rosenfeld's conjecture for cycles with just two blocks (a block being a maximal directed subpath). The conjecture was finally proved in full generality by Thomason [54] for tournaments of order $n \geq 2^{128}$. Thomason conjectured also that $N = 9$. In 2000, Havet [40] showed that $N \leq 68$ and it has been the best result so far.

Chapter 5

Long cycles in bipartite graphs

Throughout this chapter, $G = (X, Y; E)$ will denote a bipartite graph with vertex bipartition (X, Y) and the set of edges E. Let us recall that G is *balanced* if $|X| = |Y|$. A balanced bipartite graph is *bipancyclic* if it contains cycles of all even lengths from 4 up to its order.

5.1 Hamiltonicity and bipancyclicity

The aim of this section is to discuss the bipartite counterparts of the theorems introduced in Section 4.2.

All the hamiltonian criteria presented at the beginning of Section 4.2 (Theorems 4.2.1 - 4.2.4) have analogues for bipartite graphs. In 1963 Moon and Moser found Dirac-, Ore-, and Erdős-type sufficient conditions for existence of a hamiltonian cycle in a balanced bipartite graph.

Theorem 5.1.1 (Moon and Moser [48]). *Let $G = (X, Y; E)$ with $|X| = |Y| = n \geq 2$. Suppose that at least one of the following conditions is satisfied*

(1) $\delta(G) \geq (n+1)/2$,

(2) $d_G(x) + d_G(y) \geq n+1$ *for every pair of non-adjacent vertices* $x \in X$ *and* $y \in Y$,

(3) $||G|| > n(n-1) + 1$,

(4) $\delta(G) \geq r$, *where* $1 \leq r \leq n/2$, *and* $\|G\| > n(n-r) + r^2$.

Then G is hamiltonian.

Remark 5.1.2. All the bounds presented in the above theorem are best possible. Consider the balanced bipartite graph G with vertex classes of the form $X = P \cup Q$, $Y = R \cup S$, where $|P| = |R| = r$, $|Q| = |S| = n - r$, $N_G(x) = R$ for all $x \in P$, and $N_G(x) = Y$ for all $x \in Q$. Then $d_G(x) + d_G(y) \geq n$ for every pair of non-adjacent vertices $x \in X$ and $y \in Y$, $\delta(G) \geq r$ if only $r \leq n/2$, and

$\|G\| = n(n-r)+r^2$. However, as is easy to see, G is not hamiltonian. This example shows that neither the condition (2) nor (4) in Theorem 5.1.1 can be weakened. By taking $r = n/2$ (resp. $r = 1$) in the above example we show that the condition (1) (resp. (4)) cannot be improved either.

It turns out (similarly as it was in the case of simple graphs) that all the conditions presented in Theorem 5.1.1 are sufficient also for bipancyclicity, as we can see in the following four results.

Theorem 5.1.3 (Schmeichel and Mitchem [53]). *Let $G = (X, Y; E)$ be a balanced bipartite graph with $|X| = |Y| = n \geq 2$. If $\delta(G) \geq (n+1)/2$, then G is bipancyclic.*

Theorem 5.1.4 (Bagga and Varma [11]). *Let $G = (X, Y; E)$ be a balanced bipartite graph of order $2n$. If $d_G(x) + d_G(y) \geq n+1$ for every pair of non-adjacent vertices, then G is bipancyclic.*

Theorem 5.1.5 (Mitchem and Schmeichel [47]). *Let $G = (X, Y; E)$ be a balanced bipartite graph on $2n$ vertices. If $||G|| > n(n-1)+1$, then G is bipancyclic.*

Theorem 5.1.6 (Bagga and Varma [11]). *Let $G = (X, Y; E)$ be a balanced bipartite graph of order $2n$. If $\delta(G) \geq r$, where $1 \leq r \leq n/2$, and $\|G\| > n(n-r)+r^2$, then G is bipancyclic.*

Theorem 4.2.6 has its analogue in the case of bipartite graphs as well. In 1988, Entringer and Schmeichel proved the following theorem.

Theorem 5.1.7 (Entringer and Schmeichel [30]). *Let G be a hamiltonian bipartite graph of order $2n \geq 8$. If $\|G\| > n^2/2$, then G is bipancyclic.*

The authors showed in [30] that the bound $n^2/2$ may not be best possible, however it cannot be improved below $\lfloor \frac{1}{4}(n+1)(n+3) \rfloor$. Three years later Amar got the following result which has not been improved so far.

Theorem 5.1.8 (Amar [6]). *Let G be a hamiltonian bipartite graph of order $2n$. If $\|G\| > \frac{n(n+3)}{3}$, then G is bipancyclic.*

Bagga and Varma [10, 11] studied how Theorems 5.1.3 - 5.1.6 could be extend to general (not necessarily balanced) bipartite graphs . They got the following results.

Theorem 5.1.9 (Bagga and Varma [11]). *Let $G = (X, Y; E)$ be a bipartite graph with $|X| = m \leq n = |Y|$. If $||G|| \geq n(m-1)+2$, then G contains cycles C_{2p} for all $2 \leq p \leq m$.*

Theorem 5.1.10 (Bagga and Varma [11]). *Let $G = (X, Y; E)$ be a bipartite graph with $|X| = m \leq n = |Y|$. If one of the following degree conditions is satisfied*

(1) $\delta(G) \geq \frac{m+n+2}{4}$,

(2) $d_G(x) + d_G(y) \geq \frac{m+n+2}{2}$ *for every pair of non-adjacent vertices* $x \in X$ *and* $y \in Y$,

then G contains cycles C_{2p} for all $2 \leq p \leq m$.

Amar and Manoussakis [9] found a better bound on the minimal degree of G. They proved the following theorem.

Theorem 5.1.11 (Amar and Manoussakis [9]). *Let $G = (X, Y; E)$ be a bipartite graph with $|X| = m \leq n = |Y|$. If $\delta(G) \geq \frac{m+2}{2}$, then G contains cycles C_{2p} for all $2 \leq p \leq m$.*

Interestingly, Theorem 5.1.6 as well as condition (4) of Theorem 5.1.1 still do not have any extension to general bipartite graphs.

Finally, let us mention that the notion of Bondy-Chvátal closure, introduced in Section 4.2, has its analogue for bipartite graphs. Given a balanced bipartite graph $G = (X, Y; E)$, one defines a *k-biclosure* $BCl_k(G)$ of G as the graph obtained from G by recursively joining pairs of non-adjacent vertices $x \in X$ and $y \in Y$, with degree sum of at least k, until no such pair remains. Closely related to this construction is the notion of *k-bistability*: A property $\mathcal{P}$ defined on all balanced bipartite graphs of order $2n$ is called *k-bistable* when, whenever $G + xy$ has the property $\mathcal{P}$ and $d_G(x) + d_G(y) \geq k$, then G itself has the property $\mathcal{P}$.

Bondy and Chvátal proved that the property of being hamiltonian is $(n+1)$-bistable.

Theorem 5.1.12 (Bondy and Chvátal [24]). *A balanced bipartite graph G of order $2n$ is hamiltonian if and only if its $(n+1)$-biclosure $BCl_{n+1}(G)$ is so.*

For cycles of other even lengths we have the following result.

Theorem 5.1.13 (Amar, Favaron, Mago and Ordaz [7]). *The property of containing a cycle of length $2s$ is $(2n - s + 1)$-bistable on balanced bipartite graphs of order $2n$.*

The value of k for which the property $\mathcal{P}$: "G is bipancyclic" is k-bistable is unknown. For sure, $\mathcal{P}$ is not $(n+1)$-bistable. For n odd, consider the graph G of order $2n$ defined by the cycle $a_1b_1a_2b_2 \ldots a_nb_na_1$ with additional edges a_1b_j for j odd, $3 \leq j \leq n-2$, and a_ib_2 for i odd, $5 \leq i \leq n$. G has no cycles of length 4, while the graph $G + a_1b_2$ is bipancyclic and $d_G(a_1) + d_G(b_2) = n + 1$. However, we have the following theorem (which is an analogue of Theorem 4.2.8).

Theorem 5.1.14 (Amar, Favaron, Mago and Ordaz [7]). *Let G be a balanced bipartite graph of order $2n$. If the $(n+2)$-biclosure of G is the complete bipartite graph $K_{n,n}$, then G is bipancyclic.*

5.2 Size condition

Let us quote again the theorem by Bagga and Varma (Theorem 5.1.9) presented in the previous section.
Let $G = (X, Y; E)$ be a bipartite graph with $|X| = m \leq n = |Y|$. If $||G|| \geq n(m-1)+2$, then G contains cycles C_{2p} for all $2 \leq p \leq m$.

Our goal is to generalize this theorem to long cycles, that is to obtain an analogue of Woodall's theorem (Theorem 4.2.9) for bipartite graphs. We want to solve the following problem.

Problem 2. For a bipartite graph $G = (X, Y; E)$ with $|X| = m \leq n = |Y|$, and integer $k < \frac{m}{2}$, find the minimum number of edges of G, $g(m, n, k)$, which guarantees the existence of a cycle of length $2m - 2k$ in G.

Before introducing the solution, let us define the family $\mathcal{G}_{m,n,k}$ of bipartite graphs $G = (X, Y; E)$ of order $m+n$ such that $|X| = m \leq n = |Y|$, $X = U \cup W$, $|W| = k+1$, $d_G(u) = n$ for every $u \in U$, and $d_G(w) = 1$ for every $w \in W$. Note that every graph of the family $\mathcal{G}_{m,n,k}$ has size equal to $n(m-k-1)+k+1$ and does not contain any cycle of length $2m - 2k$. Moreover, if we add any edge to a graph of $\mathcal{G}_{m,n,k}$, then it will contain a cycle C_{2m-2k}.

Problem 2 is partially solved as can be seen in the following theorem. A variant of this theorem for balanced bipartite graphs was first announced in [2].

Theorem 5.2.1 (LA [4]). *Let $G = (X, Y; E)$ be a bipartite graph with $|X| = m \leq n = |Y|$, where $m \geq \frac{1}{2}k^2 + \frac{3}{2}k + 4$, $k \in \mathbb{N}$. If*

$$||G|| \geq g(m, n, k) = n(m-k-1) + k + 1,$$

then either G contains a cycle of length $2m - 2k$, or else $||G|| = g(m, n, k)$ and G is isomorphic to a graph of the family $\mathcal{G}_{m,n,k}$.
In both cases G contains C_{2p} for all $2 \leq p \leq m-k-1$.

Of course, Theorem 5.1.9 is a special case (k=0) of the above result.

Before presenting the proof of Theorem 5.2.1, we have to introduce a notion of hamiltonian biconectedness. We follow the definition used in [7].

A balanced bipartite graph $G = (X, Y; E)$ is *hamiltonian biconnected* if for any two vertices $x \in X$ and $y \in Y$, there exists a hamiltonian xy-path in G.

It is *2-hamiltonian biconnected* if for any pair of vertices $a \in X$ and $b \in Y$, the subgraph $G - \{a, b\}$ is hamiltonian biconnected.

In the course of the proof of Theorem 5.2.1, we shall use the following result.

Theorem 5.2.2 (Amar, Favaron, Mago and Ordaz [7]). *Let $G = (X, Y; E)$ be a balanced bipartite graph of order $2n$. If $||G|| \geq n^2 - n + 3$, then G is hamiltonian biconnected and 2-hamiltonian biconnected.*

Proof of Theorem 5.2.1. The proof will be divided into two steps. We first show that the theorem is true for $m = n$. In the second step we will consider the case when $m < n$.

Step 1. $m = n$

We will proceed by induction on k.

Without loss of generality we can assume that the size of G is equal to $g(n, n, k)$, for otherwise we may consider a spanning subgraph of G with exactly $g(n, n, k)$ edges instead of G.

Set $k = 0$. We want to show that if G is a balanced bipartite graph with $|G| = 2n \geq 8$ and $||G|| = n^2 - n + 1$ then G is bipancyclic unless $G \in \mathcal{G}_{n,n,0}$.

Applying Theorem 5.1.4 to the graph G, we get that either G is bipancyclic, or else there are two non-adjacent vertices $x \in X$ and $y \in Y$ such that $d_G(x) + d_G(y) \leq n$. But then $||G - \{x, y\}|| \geq n^2 - n + 1 - n = (n-1)^2$, so $G - \{x, y\}$ is a complete balanced bipartite graph on $2n - 2$ vertices (hence G contains cycles of all even lengths up to $2n - 2$). If so, then, in fact, $d_G(x) + d_G(y) = n$. Since x and y are not adjacent, then their degrees are both greater than zero. In case $d_G(x) \geq 2$ and $d_G(y) \geq 2$ it is easy to find a hamiltonian cycle in G. It remains to consider the case when one vertex, say x, has degree 1. Then the vertex y has degree $n - 1$ and it is joined with all vertices of the set $X \setminus \{x\}$. It means that $G \in \mathcal{G}_{n,n,0}$.

Now assume the theorem holds for $k - 1$, $k \geq 1$; we will prove that it holds also for k.
Let us start with the observation that $g(n, n, k) - g(n-1, n-1, k-1) = n - k$. In the proof we shall consider two cases.

Case 1. There is a pair of vertices $x \in X$ and $y \in Y$ in the graph G such that $d_G(x) + d_G(y) \leq n - k$.

Then $G - \{x, y\}$ is a balanced bipartite graph with $2n - 2$ vertices and at least $g(n, n, k) - (n - k) = g(n-1, n-1, k-1)$ edges. By the inductive hypothesis there are cycles of all even lengths up to $2(n-1) - 2(k-1) = 2n - 2k$ contained in $G - \{x, y\}$, hence also in G, unless $G - \{x, y\} \in \mathcal{G}_{n-1,n-1,k-1}$. In the latter case $d_G(x) + d_G(y) = n - k$ and x and y are not adjacent, because $||G - \{x, y\}|| = g(n-1, n-1, k-1)$. Moreover, $X \setminus \{x\} = U \cup W$, where $U =$

$\{v : d_{G-\{x,y\}}(v) = n-1\}$, $W = \{v : d_{G-\{x,y\}}(v) = 1\}$ and $|W| = k$. We see at once that cycles of all even lengths up to $2n-2k-2$ are contained in $G - \{x, y\}$, hence also in G. What remains to show is that either $C_{2n-2k} \subset G$ or $G \in \mathcal{G}_{n,n,k}$.
It is easy to check that $C_{2n-2k} \subset G$ in case $d_G(x) \geq 2$. It remains to consider what happens when $d_G(x) \leq 1$. Suppose that $d_G(x) = 0$. Then $d_G(y) = n-k > k$, which implies that there is at least one vertex in the set U, say v_1, and at least one vertex, say v_2, in the set W which are joined with the vertex y. In this case, it is a simple matter to find a cycle C_{2n-2k} contained in G which passes through the vertices v_1, y and v_2. So suppose that $d_G(x) = 1$, which implies $d_G(y) = n-k-1 > k$. Then either the vertex y is joined with all vertices in the set U (which means that $G \in \mathcal{G}_{n,n,k}$), or, again, y is joined with some vertex of U and with at least one vertex in W, and we again find a cycle of length $2n-2k$ contained in G.

Case 2. For every pair of vertices $x \in X$ and $y \in Y$ in the graph G, we have $d_G(x) + d_G(y) \geq n-k+1$.

Firstly, observe that this assumption implies that $\delta(G) \geq 2$, for otherwise G has more than $n(n-k)$ edges, which contradicts the assumption made at the beginning of Step 1 that $||G|| = g(n, n, k)$.

Suppose now that there exists $2 \leq p \leq n-k$ such that G does not contain any cycle of length $2p$.
Theorem 5.1.4 implies that there is a pair of non-adjacent vertices $x_1 \in X$ and $y_1 \in Y$ such that $d_G(x_1) + d_G(y_1) \leq n$. Let $G_1 := G - \{x_1, y_1\}$. Observe that $||G_1|| \geq g(n, n, k) - n$. Now we apply Theorem 5.1.4 to the graph G_1. Again, G_1 is not bipancyclic, so there are two non-adjacent vertices $x_2 \in X \cap V(G_1)$ and $y_2 \in Y \cap V(G_1)$ such that $d_{G_1}(x_2) + d_{G_1}(y_2) \leq n-1$. Write $G_2 := G - \{x_1, y_1, x_2, y_2\}$. We have $||G_2|| \geq ||G_1|| - (n-1) \geq g(n, n, k) - n - (n-1)$. We now apply Theorem 5.1.4 to the graph G_2, and so on. After $k+1$ steps we finally get the graph $G_{k+1} := G - \{x_1, x_2, \ldots, x_{k+1}, y_1, y_2, \ldots, y_{k+1}\}$ such that $||G_{k+1}|| \geq ||G_k|| - (n-k) \geq g(n, n, k) - n - (n-1) - \ldots - (n-k) =$ $= g(n, n, k) - \frac{2n-k}{2} \cdot (k+1) = n^2 - 2nk - 2n + \frac{1}{2}k^2 + \frac{3}{2}k + 1$.
For $n \geq \frac{1}{2}k^2 + \frac{3}{2}k + 4$, we have

$$n^2 - 2nk - 2n + \frac{1}{2}k^2 + \frac{3}{2}k + 1 \geq (n-k-1)^2 - (n-k-1) + 3,$$

and G_{k+1} is hamiltonian biconnected and 2-hamiltonian biconnected by Theorem 5.2.2, and also bipancyclic by Theorem 5.1.5. It follows that there are cycles of all even lengths up to $2n-2k-2$ contained in G_{k+1}, hence also in G. So the hypothesis, that there exists $2 \leq p \leq n-k$ such that $C_{2p} \not\subset G$, implies that G does not contain any cycle of length $2n-2k$.

Let $V' := V(G)\backslash V(G_{k+1})$.
We need to consider several subcases.

Subcase 2.1. There is a pair of vertices $x \in X \cap V'$ and $y \in Y \cap V'$ such that $e(x, V') = e(y, V') = 0$.

Since $d_G(x) + d_G(y) \geq n - k + 1$, there is a vertex $u \in Y \cap V(G_{k+1})$ which is joined with x, and there is a vertex $v \in X \cap V(G_{k+1})$ which is joined with y. There exists a uv-path of order $2n - 2k - 2$ in G_{k+1}, because G_{k+1} is hamiltonian biconnected. It means that there is a path $P_{2n-2k} = x^1y^1x^2y^2 \dots x^{2n-2k}y^{2n-2k}$ ($x^1 = x$, $y^1 = u$, $x^{2n-2k} = v$, $y^{2n-2k} = y$) contained in G, which passes through all vertices of the graph G_{k+1}. Since $C_{2n-2k} \not\subset G$, then $x^iy^{2n-2k} \notin E$, if $x^1y^i \in E$, for otherwise $x^1y^1x^2y^2 \dots x^iy^{2n-2k}x^{2n-2k} \dots y^ix^1$ is a cycle of length $2n - 2k$ contained in G. Thus, we have $e(x, V(G_{k+1})) + e(y, V(G_{k+1})) \leq n - k$, which, in this case, means that $d_G(x) + d_G(y) \leq n - k$, contradicting the assumption.

Subcase 2.2. For every pair of vertices $x \in X \cap V'$ and $y \in Y \cap V'$, we have $e(x, V') \neq 0$ or $e(y, V') \neq 0$.

We then consider an edge xy with $x \in X \cap V'$, $y \in Y \cap V'$.

Subcase 2.2.1. For at least one edge xy with $x \in X \cap V'$, $y \in Y \cap V'$, we have $e(x, V(G_{k+1})) \geq 1$ and $e(y, V(G_{k+1})) \geq 1$.
Then, again, there is a vertex $u \in Y \cap V(G_{k+1})$ which is joined with x, and there is a vertex $v \in X \cap V(G_{k+1})$ which is joined with y, and there exists a uv-path of order $2n - 2k - 2$ in G_{k+1}. We can extend this path to a cycle of length $2n - 2k$ using edges ux, xy, yv; a contradiction with the assumption that $C_{2n-2k} \not\subset G$.

Subcase 2.2.2. For every edge xy with $x \in X \cap V'$, $y \in Y \cap V'$, we have $e(x, V(G_{k+1})) = 0$ or $e(y, V(G_{k+1})) = 0$.
Without loss of generality we may assume that $e(x, V(G_{k+1})) = 0$. Then,

$$\text{for all vertices } u \in Y \cap V', \text{we have } e(u, V(G_{k+1})) \geq 2. \qquad (\star)$$

Indeed, $d_G(x) \leq k$ (because x is joined only with at most k vertices of V', which is a consequence of the construction of G_{k+1}), so, by assumption made at the beginning of Case 2, $d_G(u) \geq n - 2k + 1$. For $n \geq \frac{1}{2}k^2 + \frac{3}{2}k + 4$, it follows that $d_G(u) \geq k + 2$, which leads to $e(u, V(G_{k+1})) \geq 2$.
Since $\delta(G) \geq 2$, the vertex x has at least two neighbours in V', say y, y'. By $(\star)$, there are two different vertices $v_1, v_2 \in X \cap V(G_{k+1})$ such that $v_1y \in E$ and $v_2y' \in E$. G_{k+1} is hamiltonian, so there exists a vertex $u \in Y \cap V(G_{k+1})$ such that $v_1u \in E$. Let $a \in Y \cap V(G_{k+1})$ and $a \neq u$. Since G_{k+1} is 2-hamiltonian biconnected, $G_{k+1} - \{v_1, a\}$

is hamiltonian biconnected, so there exists a v_2u-path of order $2n-2k-4$ in $G_{k+1}-\{v_1,a\}$. We can extend this path to a cycle of length $2n-2k$ using edges v_2y', $y'x$, xy, yv_1, v_1u; a contradiction with the assumption that $C_{2n-2k} \not\subset G$.

Step 2. $m < n$

First, order the vertices of $Y = \{y^1, \dots, y^n\}$ so that $d_G(y^1) \leq \dots \leq d_G(y^n)$. Put $Y' := \{y^1, \dots, y^{n-m}\}$.

We shall consider two cases depending on the degree of the vertex y^{n-m}.

Case 1. $d_G(y^{n-m}) \geq m-k$

Then $G-Y'$ is a balanced bipartite graph on $2m$ vertices such that $||G-Y'|| \geq m(m-k) > g(m,m,k)$, as $m > k+1$. From what has already been proved in Step 1, we conclude that $G-Y' \supset C_{2p}$ for all $2 \leq p \leq m-k$, which completes the proof in this case.

Case 2. $d_G(y^{n-m}) \leq m-k-1$

Now we have

$$||G-Y'|| \geq g(m,n,k)-(n-m)(m-k-1) = g(m,m,k) \qquad (\diamond)$$

which again implies $G-Y' \supset C_{2p}$ for all $2 \leq p \leq m-k-1$, and also $G-Y' \supset C_{2m-2k}$ unless $||G-Y'|| = g(m,m,k)$ and $G-Y' \in \mathcal{G}_{m,m,k}$. Therefore it remains to consider the situation when $X = U \cup W$, where $U = \{v : d_{G-Y'}(v) = m\}$, $W = \{v : d_{G-Y'}(v) = 1\}$, $|W| = k+1$. Then every vertex of Y' has degree in G equal to $m-k-1$, since inequality $(\diamond)$ is, in fact, an equality.

Suppose now that no cycle of length $2m-2k$ is contained in G. Consequently, no vertex of Y' can be joined with both a vertex of U and a vertex of W, for otherwise we could simply construct a cycle C_{2m-2k} in G. Hence the assumption that $d_G(y^i) = m-k-1$ for all $1 \leq i \leq n-m$, implies (for $m > 2k+2$) that every vertex of Y' is joined with all vertices of U. It follows that $G \in \mathcal{G}_{m,n,k}$, which completes the proof.

□

Applying Erdős' approach to long cycles (presented in Section 4.2) in the case of bipartite graphs, we could say that we want to determine minimal integer $m_0(k)$ such that for $m > m_0(k)$ every graph $G = (X, Y; E)$ with $|X| = m \leq n = |Y|$ contains a cycle C_{2m-2k}, provided $||G|| \geq n(m-k-1)+k+2$. Theorem 5.2.1 shows that $m_0(k) \leq \frac{1}{2}k^2 + \frac{3}{2}k + 3$. The following example shows that $m_0(k) \geq 2k+1$.

Example 5.2.3. Let $G_1 = (X, Y; E)$ be a bipartite graph, with vertex classes of the form $X = Q \cup R$, $Y = S \cup T$, where $|Q| = |T| = m-k-1$, $|R| = k+1$, $|S| = n-m+k+1$, $m \leq n$. Fix a vertex y_0 in T, and let $N_{G_1}(v) = S \cup \{y_0\}$ for all $v \in Q$, and $N_{G_1}(w) = T$ for all $w \in R$. Then $\|G_1\| > n(m-k-1)+k+1$ for $k+3 \leq m \leq 2k+1$, yet G_1 contains no cycle of length greater than $2m-2k-2$.

We believe that $m_0(k) = 2k + 1$, and therefore we formulate the following conjecture.

Conjecture 5.2.4. *Let $G = (X, Y; E)$ be a bipartite graph with $|X| = m \leq n = |Y|$, where $m \geq 2k + 2$, $k \in \mathbb{N}$. If $||G|| > n(m - k - 1) + k + 1$, then G contains cycles of all even lengths up to $2m - 2k$.*

There are some results which suggest that this conjecture may be true.

It is true for $k = 0$ (Theorem 5.1.9). We will show that it is true also for $k = 1$, but we will need the following result (stronger than Theorem 5.1.5).

Theorem 5.2.5. *Let $G = (X, Y; E)$ be a balanced bipartite graph of order $2n$. If $||G|| \geq n^2 - n + 1$, then G is bipancyclic or $G \in \mathcal{G}_{n,n,0}$.*

The proof of Theorem 5.2.5 is the same as that of Theorem 5.2.1 in the case $k = 0$ of Step 1. □

Now we are ready to prove Conjecture 5.2.4 for $k = 1$.

Analysis similar to that in Step 2 of the proof of Theorem 5.2.1 shows that Conjecture 5.2.4 is true in general case ($m \leq n$) if only it is true for balanced bipartite graphs ($m = n$). Therefore it remains only to consider the situation when $m = n$.

Suppose that there exists $2 \leq p \leq n - 1$ such that G does not contain any cycle of length $2p$. Then, by Theorem 5.1.4, there are two vertices $x \in X$ and $y \in Y$ such that $d_G(x) + d_G(y) \leq n$. Hence $||G - \{x, y\}|| \geq n^2 - 2n + 3 - n = (n-1)^2 - (n-1) + 1$. By Theorem 5.2.5, $G - \{x, y\} \in \mathcal{G}_{n-1,n-1,0}$, and therefore contains cycles of all even lengths up to $2n-4$. Also, $||G - \{x, y\}|| = n^2 - 3n + 3$, so $d_G(x) + d_G(y) = n$ and x and y are non-adjacent. But the assumption $C_{2n-2} \not\subset G$ implies that $d_G(x) \leq 1$ and $d_G(y) \leq n - 2$; a contradiction. □

In 1985, Jackson [43] found the number of edges of a bipartite graph G necessary to ensure the existence of a cycle of length at least p in G. Conjecture 5.2.4, if true, would be a significant strengthening of the following theorem.

Theorem 5.2.6 (Jackson [43]). *Let $G = (X, Y; E)$ be a bipartite graph with $|X| = m \leq n = |Y|$, where $m \geq 2k + 2$, $k \in \mathbb{N}$. If $||G|| \geq n(m - k - 1) + k + 2$, then there exists $p \geq m - k$ such that $C_{2p} \subset G$.*

There is also another result which suggest that Conjecture 5.2.4 may be true. Gyárfás, Rousseau and Schelp solved an analogous problem for paths in bipartite graphs. They characterized all extremal graphs for this problem, which are similar to those of the family $\mathcal{G}_{m,n,k}$.

Theorem 5.2.7 (Gyárfás, Rousseau and Schelp [38]). *Let $G = (X, Y; E)$ be a bipartite graph with $|X| = m \leq n = |Y|$, where $m > 2k + 2$, $k \in \mathbb{N}$. If $||G|| \geq n(m - k - 1)$, then either $P_{2m-2k} \subset G$ or $||G|| = n(m - k - 1)$ and G is the exceptional graph for which $X = A \cup B$, $A = \{x \in X : d_G(x) = n\}$, $B = \{x \in X : d_G(x) = 0\}$, $|B| = k + 1$.*

5.3 Erdős-type condition

Given any extremal graph of the family $\mathcal{G}_{m,n,k}$ (defined above Theorem 5.2.1 in the previous section), whose minimal degree equals 1, we ask: How much can one decrease the bound on the size of a given graph G ensuring the existence of a cycle of length $2m-2k$, knowing that the minimal degree of G is greater than 1? We address this question in the following conjecture. We consider only the case of balanced bipartite graphs (when $m=n$).

Conjecture 5.3.1. *Let G be a balanced bipartite graph of order $2n$ and minimal degree $\delta(G) \geq r \geq 1$, where $n \geq 2k+2r$ and $k \in \mathbb{N}$. If*

$$\|G\| > n(n-k-r) + r(k+r),$$

then G contains a cycle of length $2n-2k$.

Observe that the bound on size in Conjecture 5.3.1 is sharp, as can be seen in the example below.

Example 5.3.2. Let G_2 be a balanced bipartite graph, with vertex classes X and Y, $|X| = |Y| = n$, where $X = Q \cup R$, $Y = S \cup T$, $|Q| = n-k-r$, $|R| = k+r$, $|S| = n-r$, and $|T| = r$. Moreover, assume that $N_{G_2}(x) = Y$ for all $x \in Q$, and $N_{G_2}(x) = T$ for all $x \in R$. Then $\delta(G_2) = r$ if only $n-k-r \geq r$, and $\|G_2\| = n(n-k-r) + r(k+r)$. However, as is easy to see, G_2 contains no cycle of length $2n-2k$.

Conjecture 5.3.1 states an Erdős-type condition for existence of long cycles in balanced bipartite graphs and holds true for $k=0$ (then it is exactly the fourth criterion for hamiltonicity of Theorem 5.1.1). In the case of $k=r=1$ Conjecture 5.3.1 was proved in the previous section (since it is a special case of Conjecture 5.2.4). The following theorem shows that Conjecture 5.3.1 also holds for $k=1$ and arbitrary r. The conjecture in the general case seems to be hard.

Theorem 5.3.3 (LA [1]). *Let $G = (X, Y; E)$ be a balanced bipartite graph of order $2n$ and minimal degree $\delta(G) \geq r \geq 1$, where $n \geq 4$ and $n \geq 2r+1$. Let*

$$g_1(n,r) = n(n-1-r) + r(1+r) + 1.$$

Then G contains a cycle of length $2n-2$, provided $\|G\| \geq g_1(n,r)$.

Remark 5.3.4. The assumption on n is weakest possible. In the case of $n = 2r$ the minimum size which guarantees the existence of a cycle C_{2n-2} in G is $2r^2+r+1$ (see Theorem 3.6 in [1]). For $n \leq 2r-1$ we do not need any condition on size, because Theorem 5.1.4 gives bipancyclicity.

Apart from results presented in Section 5.1, in the proof of Theorem 5.3.3 we will need two additional results. The first one is another hamiltonicity criterion given by Moon and Moser.

Theorem 5.3.5 (Moon and Moser, [48]). *Let $G = (X, Y; E)$ be a balanced bipartite graph of order $2n$, and let $S_a = \{x \in X : d_G(x) \leq a\}$, $T_a = \{y \in Y : d_G(y) \leq a\}$ for $a \in \mathbb{N}$. If, for every $1 \leq a \leq n/2$, the sets S_a and T_a are of cardinalities less than a, then G is hamiltonian.*

The next result is a strengthening of the fourth condition of Theorem 5.1.1.

Theorem 5.3.6 (Wojda and Woźniak, [59]). *Let G be a balanced bipartite graph of order $2n \geq 4$, minimal degree $\delta(G) \geq r \geq 1$, and size $\|G\| \geq n(n-r)+r^2$. Let $G(n, r)$ denote a bipartite graph with colour classes $X = P \cup Q$ and $Y = R \cup S$ such that $|P| = |R| = r$, $|Q| = |S| = n - r$, $N_{G(n,r)}(x) = R$ for all $x \in P$, and $N_{G(n,r)}(x) = Y$ for all $x \in Q$. Then G contains a hamiltonian cycle, else $r \leq n/2$ and G is isomorphic to $G(n, r)$.*

We are now in position to prove Theorem 5.3.3. Throughout the proof we will frequently refer to the exceptional graph $G(n, r)$ of Theorem 5.3.6.
We shall first show the following lemma.

Lemma 5.3.7. *Let $G = (X, Y; E)$ be a balanced bipartite graph of order $2n$ and minimal degree $\delta(G) \geq r \geq 1$, where $n \geq 4$ and $n \geq 2r + 1$. Let $\|G\| \geq g_1(n, r) = n(n-1-r)+r(1+r)+1$, and assume there exists a pair of vertices $x \in X$ and $y \in Y$ such that $d_G(x) + d_G(y) \leq n$ and $\delta(G - \{x, y\}) \geq r$. Then G contains a cycle of length $2n - 2$.*

Proof. Suppose G contains no cycle of length $2n-2$. Then $G-\{x, y\}$ contains no such cycle either, and as $\delta(G - \{x, y\}) \geq r$, Theorem 5.3.6 implies that

$$\|G - \{x, y\}\| \leq (n - 1)(n - 1 - r) + r^2 = n^2 - 2n - nr + r^2 + r + 1.$$

On the other hand,

$$\|G - \{x, y\}\| \geq g_1(n, r) - (d_G(x) + d_G(y)) \geq n^2 - 2n - nr + r^2 + r + 1.$$

Hence $d_G(x) + d_G(y) = n$, x and y are non-adjacent, and $G - \{x, y\}$ equals $G(n - 1, r)$. Without loss of generality, we may assume that x belongs to the vertex class of G containing $P \cup Q$ of $G(n - 1, r)$.

Now, either $d_G(x) \geq r + 1$ or $d_G(x) = r$. In the first case, x must have at least two neighbours in S or else at least one neighbour in both S and R. One easily verifies that then G contains a cycle of length $2n - 2$, omitting y and a single vertex of P; a contradiction.

If, in turn, $d_G(x) = r$, then $d_G(y) = n - r$ and y must have neighbours in both P and Q. Consequently, G contains a cycle of length $2n - 2$, ommiting x and a vertex of S, which again contradicts the choice of G. $\square$

Proof of Theorem 5.3.3. For a proof by contradiction, consider a graph G satisfying the assumptions of Theorem 5.3.3, that does not contain a cycle of length $2n-2$. Observe first that $\|G\| > n^2/2$. Indeed, the difference $g_1(n,r) - n^2/2$ is always positive. Hence, by Theorem 5.1.7, G is not hamiltonian. Consequently, Theorem 5.3.5 implies that there exists a positive integer $a \le n/2$ such that at least one of the sets $S_a = \{x \in X : d_G(x) \le a\}$, $T_a = \{y \in Y : d_G(y) \le a\}$ has cardinality greater than or equal to a.

Let l be the least such a. Without loss of generality, we may assume that l is realized in X; i.e., $|\{x \in X : d_G(x) \le l\}| \ge l$. Order the vertices of $X = \{x_1, \dots, x_n\}$ so that $r \le d_G(x_1) \le \cdots \le d_G(x_n)$. Then, by minimality of l, we have $l = \min\{i : d_G(x_i) \le i\}$. Of course, $r \le l \le n/2$. Put $L = \{x_1, \dots, x_l\}$.

The rest of the proof proceeds in two cases, depending on l being equal to or greater than r.

Case 1. $l = r$.

We will first show that all the vertices of Y have degrees greater than r. Suppose to the contrary that there exists $y_1 \in Y$ with $d_G(y_1) = r$. Then

$$\|G - \{x_1, y_1\}\| \ge g_1(n,r) - 2r = n^2 - n - nr + r^2 - r + 1,$$

and $\delta(G - \{x_1, y_1\}) \ge r - 1$. On the other hand, by Theorem 5.3.6,

$$\|G - \{x_1, y_1\}\| \le (n-1)(n-r) + (r-1)^2 = n^2 - n - nr + r^2 - r + 1.$$

Hence $d_G(x_1) + d_G(y_1) = 2r$, so that $x_1y_1 \notin E$ and $G - \{x_1, y_1\}$ equals $G(n-1, r-1)$. By comparison of degrees, one readily verifies that x_1 belongs to that vertex class of G that contains $P \cup Q$ of $G(n-1, r-1)$; in fact, $L = \{x_1\} \cup P$. Consider the sets R and S of the other vertex class of $G(n-1, r-1)$. As $|N_G(x_1)| = r > |R|$ and $x_1y_1 \notin E$, it follows that either x_1 has neighbours in both R and S or else it has at least two neighbours in S. In any case, as in the proof of Lemma 5.3.7, one easily finds a cycle of length $2n-2$ in G, ommiting y_1 and a vertex of P; a contradiction. Thus $d_G(y) \ge r + 1$ for every $y \in Y$.

Next observe that every vertex of Y has a neighbour in L. Suppose otherwise, and let $y_1 \in Y$ be such that $N_G(y_1) \subset X \setminus L$. Notice that all vertices of $X \setminus L$ have degrees greater than r, for otherwise $g_1(n,r) \le \|G\| \le (r+1)r + (n-r-1)n = g_1(n,r) - 1$. Consequently, by removing y_1 and a vertex of L, say x_1, we do not decrease the minimal degree in the remainder of G. But, as $N_G(y_1) \subset X \setminus L$, we have $d_G(y_1) \le n - r$, hence $d_G(x_1) + d_G(y_1) \le r + (n-r) = n$, and by Lemma 5.3.7, G contains a cycle of lenth $2n-2$; a contradiction.

Consider the graph $G - L$. Notice that

$$\|G - L\| \ge g_1(n,r) - r^2 = n^2 - n - nr + r + 1.$$

Moreover, we claim that $d_{G-L}(x) + d_{G-L}(y) \geq n$ for every pair of non-adjacent $x \in X \setminus L$ and $y \in Y$. For if $d_{G-L}(x) + d_{G-L}(y) \leq n - 1$ for a pair of non-adjacent $x \in X \setminus L$ and $y \in Y$, then, by the above inequality,

$$\|(G - L) - \{x, y\}\| \geq n^2 - 2n - nr + r + 2 > (n - r - 1)(n - 1),$$

which contradicts $(G - L) - \{x, y\}$ being a bipartite graph with vertex classes of cardinalities $n - r - 1$ and $n - 1$.

Taking into account that every vertex in Y has a neighbour in L, we now obtain that

$$d_G(x) + d_G(y) \geq n + 1 \quad \text{for all non-adjacent } y \in Y \text{ and } x \in X \setminus L.$$

Let $\widetilde{G}$ be the bipartite graph obtained from G by joining all the non-adjacent vertices of Y and $X \setminus L$. As $|X \setminus L| = n - r$ and every $y \in Y$ has a neighbour in L, we get that $d_{\widetilde{G}}(y) \geq n - r + 1$ for all $y \in Y$. Hence $d_{\widetilde{G}}(x) + d_{\widetilde{G}}(y) \geq n + 1$ for every pair of non-adjacent vertices $x \in X$ and $y \in Y$. Therefore, joining all the non-adjacent vertices of X and Y in $\widetilde{G}$ with degree sum of at least $n + 1$ yields a complete bipartite graph $K_{n,n}$. As $\widetilde{G}$ was obtained from G also by joining certain non-adjacent vertices of X and Y of degree sum of at least $n + 1$, this shows that the $(n + 1)$-biclosure of G equals $K_{n,n}$. Thus, by Theorem 5.1.12, G contains a hamiltonian cycle, which, as we observed at the begining of this proof, is impossible.

Case 2. $l \geq r + 1$.

In this case $n \geq 2r + 2$ (as $l \leq n/2$) and $r \geq 2$ (for otherwise $l = r = 1$, by minimality); hence $|L| \geq 3$. Moreover, $d_G(x_{l-1}) = d_G(x_l) = l$, by minimality of l.

We will first show that, for every pair of non-adjacent $x \in X \setminus L$ and $y \in Y$, $d_G(x) + d_G(y) \geq n + 2$, provided $n \geq 2r + 4$.
Put $f(l) = l^2 + (n - l - 1)(n - 1) + n + 2$. It suffices to show that $g_1(n, r) \geq f(l)$ (hence also $\|G\| \geq f(l)$). Indeed, if G contains a pair of non-adjacent vertices $x \in X \setminus L$ and $y \in Y$ with $d_G(x) + d_G(y) \leq n + 1$, then

$$\|G\| \leq |L| \cdot l + |X \setminus (L \cup \{x\})| \cdot |Y \setminus \{y\}| + d_G(x) + d_G(y) \leq f(l) - 1.$$

As the derivative of f equals $f'(l) = -n + 2l + 1$, it follows that $f(l)$ is decreasing for $l \leq (n - 1)/2$, and hence maximal at $l = r + 1$. One immediately verifies that $f(r + 1) \leq g_1(n, r)$ for $n \geq 2r + 4$. If, on the other hand, $l > (n - 1)/2$, then $l = n/2$ (since $l \leq n/2$ anyway), and it is again immediate to check that $f(n/2) \leq g_1(n, r)$ for $n \geq 2r + 4$.

Let G' be the bipartite graph obtained from G by joining all the non-adjacent vertices of $X \setminus L$ and Y. We claim that every $y \in Y$ has a neighbour in L (in G'). Suppose otherwise, and let $y_1 \in Y$ be such that $N_{G'}(y_1) \subset X \setminus L$. Then $d_{G'}(y_1) \leq n-l$, $x_1y_1 \notin E$, and $d_{G'}(x_1)+d_{G'}(y_1) \leq n$. Moreover, $\delta(G'-\{x_1, y_1\}) \geq r$, as all the vertices in $X \setminus L$ have degrees of at least $l = r+1$, and $d_{G'}(y) \geq n-l \geq l = r+1$ for all $y \in Y$. Then Lemma 5.3.7 implies that G' contains a cycle of length $2n-2$, and hence, by Theorem 5.1.13, so does G; a contradiction.

Notice that G' was obtained from G by joining only pairs of vertices with degree sum of at least $n+2$. Also, as every vertex $y \in Y$ has a neighbour in L (in G'), we have $d_{G'}(y) \geq n-l+1$. Recall that $d_{G'}(x_l) = d_G(x_l) = l$ and $d_{G'}(x_{l-1}) = d_G(x_{l-1}) = l$, hence

$$d_{G'}(x_l)+d_{G'}(y) \geq n+1 \quad \text{and} \quad d_{G'}(x_{l-1})+d_{G'}(y) \geq n+1 \quad \text{for all } y \in Y.$$

Let $G^{(2)}$ be the graph obtained from G' by joining x_l and x_{l-1} with all the vertices of Y. Then $d_{G^{(2)}}(y) \geq n-l+2$ for all $y \in Y$, and as $d_{G^{(2)}}(x_{l-2}) = d_G(x_{l-2}) \geq l-1$ (by minimality of l), we get that

$$d_{G^{(2)}}(x_{l-2}) + d_{G^{(2)}}(y) \geq n+1 \quad \text{for all } y \in Y.$$

Let now $G^{(3)}$ be the graph obtained from $G^{(2)}$ by joining x_{l-2} with all the non-adjacent vertices of Y. In general, let $G^{(m)}$ ($m \geq 3$) be obtained from $G^{(m-1)}$ by joining x_{l-m+1} with all the non-adjacent vertices of Y. Then $G^{(l)} = K_{n,n}$, and $G^{(m)}$ is obtained from $G^{(m-1)}$ by joining only pairs of vertices with degree sum of at least $n+1$. Thus $G^{(l)} = BCl_{n+1}(G)$, so that the $(n+1)$-biclosure of G is a complete bipartite graph. Theorem 5.1.12 implies that then G contains a hamiltonian cycle, which again leads to contradiction.

To complete the proof, it remains to consider the case when $n = 2r+2$ or $n = 2r+3$, and there is a pair of non-adjacent $x^0 \in X \setminus L$ and $y^0 \in Y$ with $d_G(x^0) + d_G(y^0) \leq n+1$.

Subcase 2.1. $n = 2r+2$.

Then $r+1 \leq l \leq n/2$ yields $l = r+1$. We then have

$$\left\|G - \{x^0, y^0\}\right\| \geq g_1(2r+2, r) - (2r+3) = 3r^2 + 3r, \tag{1}$$

and, on the other hand,

$$\left\|G - \{x^0, y^0\}\right\| \leq |L| \cdot l + |X \setminus (L \cup \{x^0\})| \cdot |Y \setminus \{y^0\}| = 3r^2 + 3r + 1\,. \tag{2}$$

Hence

$$3r^2 + 3r \leq \left\|G - \{x^0, y^0\}\right\| \leq 3r^2 + 3r + 1 \quad \text{and}$$
$$2r+2 \leq d_G(x^0) + d_G(y^0) \leq 2r+3\,.$$

Suppose first that $\|G - \{x^0, y^0\}\| = 3r^2+3r+1$. Then, by (2), $d_G(x) = l$ for all $x \in L$, and $N_G(y^0) \cap L = \emptyset$, hence $d_G(x_1) + d_G(y^0) \leq l+(n-l) = n$.

Moreover, $N_G(y) \supset X \setminus (L \cup \{x^0\})$ for all $y \in Y \setminus \{y^0\}$, and $d_G(x) \geq r+1$ for all $x \in X$, so that $\delta(G - \{x_1, y^0\}) \geq r$, and by Lemma 5.3.7, G contains a cycle of length $2n-2$; a contradiction.

Therefore we may assume that $\|G - \{x^0, y^0\}\| = 3r^2 + 3r$. By (1), $d_G(x^0) + d_G(y^0) = 2r+3$, and what's more, $d_G(x) + d_G(y) \geq 2r+3$ for all non-adjacent $x \in X \setminus L$ and $y \in Y$. Indeed, if $d_G(x^1) + d_G(y^1) \leq 2r+2$ for some non-adjacent $x^1 \in X \setminus L$ and $y^1 \in Y$, then by (1) and (2), $\|G - \{x^1, y^1\}\| = 3r^2 + 3r + 1$, which leads to contradiction, as above.

We will now show that $|N_G(L)| > r+1$. Suppose otherwise, that is, suppose $|N_G(L)| = l = r+1$. Then $N_G(y^0) \cap L = \emptyset$, for else $N_G(L) \ni y^0$ implies $\|G - \{x^0, y^0\}\| \leq |L| \cdot (l-1) + |X \setminus (L \cup \{x^0\})| \cdot |Y \setminus \{y^0\}| = 3r^2 + 2r$; a contradiction. Therefore $d_G(y^0) = r$, so that $d_G(x_1) + d_G(y^0) \leq l + r < n$. Notice that, as $G - \{x^0, y^0\}$ has only one edge less than the right-hand side of (2), every neighbour of y^0 in G has degree at least $n-2 = 2r$, and every neighbour of x_1 has at least $l-1 = r$ other neighbours in L (x_1 being the only vertex whose degree could be less than l). Thus $\delta(G - \{x_1, y^0\}) \geq r$, and we get a contradiction, by Lemma 5.3.7. Thus $|N_G(L)| > r+1$.

It is now not difficult to see that $BCl_{n+1}(G) = K_{n,n}$: Recall that we have verified that $d_G(x) + d_G(y) \geq 2r+3 = n+1$ for all non-adjacent $x \in X \setminus L$ and $y \in Y$. Let G' be the graph obtained from G by joining all the non-adjacent vertices of $X \setminus L$ and Y. Next observe that, by minimality of $l = r+1$, $d_{G'}(x_{r+1}) = d_G(x_{r+1}) = r+1$, and as $|N_G(L)| > r+1$, at least one non-neighbour of x_{r+1}, say y', has a neighbour among the other vertices of L. Hence $|N_{G'}(y')| \geq |X \setminus L| + 1$, so that $d_{G'}(x_{r+1}) + d_{G'}(y') \geq (r+1) + (r+2) = n+1$. Let $G^{(2)}$ be obtained from G' by joining x_{r+1} with y', and hence increasing the degree of x_{r+1} to $r+2$. Then $d_{G^{(2)}}(x_{r+1}) + d_{G^{(2)}}(y) \geq n+1$ for all $y \in Y$. Let $G^{(3)}$ be obtained from $G^{(2)}$ by joining x_{r+1} with all the non-adjacent vertices of Y. Now $d_{G^{(3)}}(y) \geq r+2$ for all $y \in Y$. By minimality of l again, $d_{G^{(3)}}(x_r) = d_G(x_r) = r+1$, and hence $d_{G^{(3)}}(x_r) + d_{G^{(3)}}(y) \geq 2r+3$ for all $y \in Y$. Let $G^{(4)}$ be obtained from $G^{(3)}$ by joining x_r with all the non-adjacent vertices of Y. Then $d_{G^{(4)}}(y) \geq r+3$ for all $y \in Y$, and hence, as $\delta(G^{(4)}) \geq \delta(G) \geq r$, $d_{G^{(4)}}(x) + d_{G^{(4)}}(y) \geq 2r+3$ for all non-adjacent $x \in X$ and $y \in Y$. Joining all the non-adjacent pairs $x \in X$, $y \in Y$ of $G^{(4)}$ with degree sum of at least $n+1$ we thus obtain $K_{n,n}$. Since at each stage we only joined pairs of vertices with degree sum of at least $n+1$, this shows that $K_{n,n} = BCl_{n+1}(G)$. By Theorem 5.1.12, G contains a hamiltonian cycle; a contradiction.

Subcase 2.2. $n = 2r+3$.

Again, $r+1 \leq l \leq n/2$ yields $l = r+1$, and we have

$$\left\|G - \{x^0, y^0\}\right\| \geq g_1(2r+3, r) - (2r+4) = 3r^2 + 6r + 3\,,$$

and, on the other hand,

$$\left\|G - \{x^0, y^0\}\right\| \leq |L| \cdot l + |X \setminus (L \cup \{x^0\})| \cdot |Y \setminus \{y^0\}| = 3r^2 + 6r + 3\,.$$

Therefore both inequalities must, in fact, be equalities, hence, in particular, $d_G(x_1) = l$ and $d_G(x) \geq r+1$ for all $x \in X$, $N_G(y^0) \cap L = \emptyset$, so that $d_G(y^0) \leq n - l$, and finally $|N_G(y)| \geq |X \setminus (L \cup \{x^0\})| = r+1$ for all $y \in Y \setminus \{y^0\}$. Thus, again, G with the vertices x_1, y^0 satisfies the assumptions of Lemma 5.3.7, hence G contains a cycle of length $2n-2$; a contradiction. This completes the proof of Theorem 5.3.3.

$\square$

Chapter 6

Orientations of long cycles in digraphs

6.1 Problem for general digraphs

Let us quote again Corollary 4.3.15 presented in Section 4.3.

Let D be a digraph of order $n \geq 9$ with $||D|| \geq (n-1)^2$. Then D contains every orientation of a cycle of length p for all $3 \leq p \leq n$ unless D is the digraph consisting of K^*_{n-1} plus a new vertex which dominates or is dominated by all the vertices of the K^*_{n-1}.

We would like to generalize this result to long cycles, that is to obtain an analogue of Woodall's theorem (Theorem 4.2.9) for digraphs. We want to solve the following problem.

Problem 3. For any pair of nonnegative integers n and $k < \frac{n}{2}$, find the minimum integer $h(n,k)$, such that every digraph of order n and size greater than $h(n,k)$ contains every orientation of a cycle of length $n-k$.

To obtain the solution of this problem, which is Corollary 6.1.6, we first prove Theorem 6.1.1 about symmetric and almost symmetric cycles in digraphs.

Before stating the theorem, we have to define some special families of digraphs. Let D_1 and D_2 be vertex disjoint digraphs.
By $D_1 \rightleftharpoons D_2$ we denote the family of digraphs D such that $V(D) = V(D_1) \cup V(D_2)$ and for every $x, y \in V(D)$: $(x \to y) \in A(D)$ if and only if

(1) $x, y \in V(D_i)$ and $(x \to y) \in A(D_i)$ $(i = 1, 2)$ or

(2) $x \in V(D_i), y \in V(D_j), i \neq j$ and $(y \to x) \notin A(D)$ (the vertices of $V(D_1)$ are joined with the vertices of $V(D_2)$ by antisymmetric arcs).

Among the digraphs of the family $D_1 \rightleftharpoons D_2$ we distinguish one: $D_1 \to D_2$ with the arc set consisting of the arcs of D_1 and D_2, and all arcs from D_1 to D_2. Let us denote by $\mathcal{H}_{n,k}$ the family $K^*_{n-k-1} \rightleftharpoons K^*_{k+1}$.

Finally, let $D_1 * D_2$ denote the digraph with $V(D_1 * D_2) = V(D_1) \cup V(D_2)$ and the arc set consisting of the arcs of D_1 and D_2, and all arcs between D_1 and D_2.

Now we are ready to present the theorem about symmetric and almost symmetric cycles in digraphs.

Theorem 6.1.1 (LA and Wojda [5]). *Let D be a digraph on n vertices, where $n \geq 7$, $n \geq \frac{5}{2}k+6$, and with at least $h(n,k) = (n-k-1)(n-1)+k(k+1)$ arcs. Then:*

(1) D contains symmetric cycles C_p^* for all $3 \leq p \leq n-k-2$,

(2) D contains an almost symmetric cycle $C_{n-k-1}^{*'}$; moreover, if $n \geq 3k+6$ then D contains a symmetric cycle C_{n-k-1}^*,

(3) D contains an almost symmetric cycle $C_{n-k}^{*'}$ unless $||D|| = h(n,k)$ and

(3a) D is one of the digraphs from $\mathcal{H}_{n,k}$,

(3b) $n = 3k+4$ and $D = T_{2k+3}^{\rightleftharpoons} * K_{k+1}^*$,

(3c) $n = 3k+2$ and $D = T_{2k+2}^{\rightleftharpoons} * K_k^*$,

where $T_p^{\rightleftharpoons}$ is a tournament of order p.

In the proof of the above theorem we will use two results presented in Section 4.2: Theorems 4.2.4 and 4.2.9. We will apply them to the special undirected graph $G(D)$, so called *associated graph* with a digraph D, with the same vertex set as D such that the edges of $G(D)$ correspond to the symmetric arcs of D. We shall make occasional use of the obvious fact that $||G(D)|| \geq \binom{n}{2} - m$ for a digraph D with n vertices, whenever $||\overline{D}|| \leq m$.

Proof of Theorem 6.1.1 (1). Observe that for digraphs with n vertices and at least $h(n,k)$ arcs we have $||\overline{D}|| \leq (k+1)(n-k-1)$. Hence

$$||G(D)|| \geq \binom{n}{2} - (k+1)(n-k-1) = \frac{(n-k-2)(n-k-1)+k(k+1)}{2}.$$

For $n \geq \frac{5}{2}k+6$, have

$$\frac{(n-k-2)(n-k-1)+k(k+1)}{2} \geq \binom{n-k-3}{2} + \binom{k+4}{2} + 1,$$

so can apply Theorem 4.2.9 to the graph $G(D)$ (take the function $f(n,k+2)$), which completes the proof. $\square$

Proof of Theorem 6.1.1 (2). The assertion is a consequence of Lemmas 6.1.2 and 6.1.3, and the already proved part (1) of Theorem 6.1.1.

Applying Theorem 4.2.9, as above, to the graph $G(D)$ (with $f(n, k+1)$) we get the following lemma.

Lemma 6.1.2. *Let D be a digraph with $|D| = n \geq 3k+6$ and $||D|| \geq h(n,k)$. Then D contains symmetric cycles C_p^* for all $3 \leq p \leq n-k-1$.* □

Lemma 6.1.3. *Let D be a digraph with $|D| = n \geq 2k+4$ and $||D|| \geq h(n,k)$. Suppose that a symmetric cycle C_{n-k-2}^* is contained in D. Then D contains an almost symmetric cycle $C_{n-k-1}^{*'}$.*

Proof. Let D be a digraph satisfying the assumptions of the lemma. We can partition the set of vertices of D into two subsets: vertices of C_{n-k-2}^* and the remaining ones, which define a set denoted by V_{k+2}.

Suppose that a cycle $C_{n-k-1}^{*'}$ is not contained in D. It implies that any vertex from V_{k+2} cannot be joined to any two consecutive vertices of the cycle C_{n-k-2}^* by more than two arcs. It means that there are at most $(n-k-2)(k+2)$ arcs between the vertices of C_{n-k-2}^* and V_{k+2}. Hence $||D|| \leq (n-k-3)(n-k-2) + (n-k-2)(k+2) + (k+1)(k+2)$. On the other hand, $||D|| \geq h(n,k) = (n-k-1)(n-1) + k(k+1)$. These inequalities hold simultaneously only when $n \leq 2k+3$, which contradicts our assumption. □

Proof of Theorem 6.1.1 (3). The proof is by induction on k.

Set $k = 0$. In this case we have to show that if D is a digraph with $|D| = n \geq 7$ and $||D|| \geq (n-1)^2$ then an almost symmetric hamiltonian cycle is contained in D, unless $D \in \mathcal{H}_{n,0}$.

We can apply Theorem 4.2.4 to the graph $G(D)$ if every vertex in $G(D)$ has its degree greater than or equal to 2, because

$$\max\left\{\binom{n-r}{2} + r^2, \binom{n-\lfloor\frac{n-1}{2}\rfloor}{2} + \left\lfloor\frac{n-1}{2}\right\rfloor^2\right\} \leq \binom{n-1}{2} \text{ for } r=2 \text{ and } n \geq 7,$$

as one can compute easily. If there is a vertex v of degree at most 1 in $G(D)$, we have two possibilities: either $G(D)$ is a graph consisting of a complete graph K_{n-1} and one isolated vertex v, or $G(D)$ is a graph consisting of an almost complete graph $K_{n-1}^{'}$ (K_{n-1} minus one edge) and one pendent vertex v. In the first case, D is a digraph of family $\mathcal{H}_{n,0}$. In the second case, D contains an almost symmetric hamiltonian cycle, as is easy to check.

Now assume the theorem holds for $k-1$, $k \geq 1$, we will prove it for k. Let us start with the observation that $h(n,k) - h(n-1,k-1) = n+k-1$. In the proof we shall consider 2 cases.

Case 1. There is a vertex in the digraph D such that $d_D(v) \leq n+k-1$.

Then the digraph $D \setminus \{v\}$ has $n-1$ vertices and at least $h(n-1,k-1)$ arcs. By inductive assumption there is an almost symmetric cycle $C^{*'}_{(n-1)-(k-1)} = C^{*'}_{n-k}$ contained in $D \setminus \{v\}$, hence also in D, unless $D \setminus \{v\}$ is one of the exceptional digraphs. If so, then $d_D(v) = n+k-1$, because $||D \setminus \{v\}|| = h(n-1,k-1)$.

Subcase 1.1. $D \setminus \{v\}$ is a digraph of the family $\mathcal{H}_{n-1,k-1}$, so the digraph D consists of complete digraphs K^*_{n-k-1}, K^*_k, and a vertex v, such that the vertices of K^*_{n-k-1} and K^*_k are joined by $(n-k-1)k$ antisymmetric arcs.

Suppose that D contains no almost symmetric cycle $C^{*'}_{n-k}$. Suppose that a vertex v is joined by a symmetric arc with a vertex of K^*_{n-k-1}. Then there are no other arcs between v and the vertices of K^*_{n-k-1} and no symmetric arcs between v and the vertices of K^*_k. Hence $d_D(v) \leq 2+k$. But $d_D(v) = n+k-1$, which is possible only for $n \leq 3$. This contradicts our assumption. So there are no symmetric arcs between the vertex v and the vertices of K^*_{n-k-1}, and since $d_D(v) = n+k-1$, v is joined by symmetric arcs with all vertices of K^*_k and by antisymmetric arcs with all vertices of K^*_{n-k-1}. Thus $D \in \mathcal{H}_{n,k}$.

Subcase 1.2. $D \setminus \{v\} = T^{\rightleftharpoons}_{2k+1} * K^*_k$, $n-1 = 3(k-1)+4$ and $d_D(v) = 4k+1$.

If v is joined with every vertex of K^*_k by a symmetric arc, then either D is the exceptional digraph $T^{\rightleftharpoons}_{2k+2} * K^*_k$, or v is joined with at least one vertex, say x, of $T^{\rightleftharpoons}_{2k+1}$ by a symmetric arc. Choose a vertex $y \in V(T^{\rightleftharpoons}_{2k+1})$ which is joined with v. Then we can extend a symmetric path $P^*_{2k+1} = xa_1b_1 \dots a_{k-1}b_{k-1}a_ky$, where $a_i \in V(K^*_k)$ for $i = 1, \dots, k$, $b_j \in V(T^{\rightleftharpoons}_{2k+1})$ for $j = 1, \dots, k-1$, to an almost symmetric cycle $C^{*'}_{2k+2}$ using the vertex v.
If there is a vertex in K^*_k which is not joined with v by a symmetric arc, then the degree of v implies existence of a symmetric arc between v and a vertex of $T^{\rightleftharpoons}_{2k+1}$ and we can repeat the above arguments for constructing $C^{*'}_{2k+2}$ in D.

Subcase 1.3. $D \setminus \{v\} = T^{\rightleftharpoons}_{2k} * K^*_{k-1}$, $n-1 = 3(k-1)+2$ and $d_D(v) = 4k-1$.

Of course, there are at most $2(k-1)$ arcs between v and the vertices of K^*_k, and since $d_D(v) = 4k-1$, there are at least $4k-1-(2k-2) = 2k+1$ arcs between v and the vertices of $T^{\rightleftharpoons}_{2k}$. It means that there is a symmetric arc between v and a vertex, say x, of $T^{\rightleftharpoons}_{2k}$. As in the former subcase, we can extend a symmetric path $P^*_{2k-1} = xa_1b_1 \dots a_{k-1}b_{k-1}$, where $a_i \in V(K^*_{k-1})$, $b_i \in V(T^{\rightleftharpoons}_{2k})$ for $i = 1, \dots, k-1$, to an almost symmetric cycle of length $2k$ using the vertex v.

Case 2. All vertices of the digraph D have degrees of at least $n+k$. We can prove two simple facts under this assumption.

Claim 1. *If D contains a symmetric path P^*_{n-k}, then D contains an almost symmetric cycle $C^{*'}_{n-k}$.*

Proof of Claim 1. Set $P^*_{n-k} = x_1x_2\dots x_{n-k}$. Suppose that D does not contain $C^{*'}_{n-k}$. Then

$$a_D(x_1, x_{i+1}) + a_D(x_{n-k}, x_i) \le 2, \quad i = 1, 2, \dots, n-k-1.$$

Hence $d_D(x_1) + d_D(x_{n-k}) \le 2(n-k-1) + 4k$, a contradiction. □

Claim 2. *If D has at most $3k+3$ vertices, D contains a symmetric path P^*_{n-k-1} and does not contain any symmetric path P^*_{n-k}, then D contains C^*_{n-k-1}.*

Proof of Claim 2. The condition $d_D(v) \ge n+k$ for every vertex v of D implies that $\delta(G(D)) \ge k+1$. Now suppose that $G(D)$ does not contain any cycle C_{n-k-1} (it means that $C^*_{n-k-1} \not\subset D$). Let $P_{n-k-1} = x_1x_2\dots x_{n-k-1}$ be a path of $G(D)$. Then $e_G(x_1, x_{i+1}) + e_G(x_{n-k-1}, x_i) \le 1$ for every $i = 1, 2, \dots, n-k-2$ (where $e_G(u, v)$ denotes the number of edges joining the vertices u and v in $G(D)$). Thus $d_G(x_1) + d_G(x_{n-k-1}) \le n-k-2$ (since $G(D)$ does not contain P_{n-k}). But $d_G(x_1) + d_G(x_{n-k-1}) \ge 2(k+1)$. Both inequalities imply $n \ge 3k+4$, contrary to the assumption of the claim. □

We will need the following two lemmas.

Lemma 6.1.4. *Let D be a digraph with $|D| = n$ and $||D|| \ge h(n,k)$. Suppose that a symmetric cycle C^*_{n-k-1} is contained in D. Then an almost symmetric cycle $C^{*'}_{n-k}$ is contained in D unless D belongs to the family $\mathcal{H}_{n,k}$.*

Proof. Let D be a digraph which satisfies the conditions of the lemma. The vertices which are not in C^*_{n-k-1} define a set V_{k+1}.

Suppose that $C^{*'}_{n-k}$ is not contained in D. Then there are at most $(n-k-1)(k+1)$ arcs between the vertices of C^*_{n-k-1} and V_{k+1}. But $h(n,k) - (n-k-1)(k+1) = (n-k-1)(n-1) + k(k+1) - (n-k-1)(k+1) = = (n-k-1)(n-k-2) + k(k+1)$, hence the subdigraphs induced by the vertices of C^*_{n-k-1} and V_{k+1} are complete digraphs K^*_{n-k-1} and K^*_{k+1}, respectively. We conclude that the digraph D has exactly $(n-k-1)(k+1)$ antisymmetric arcs between K^*_{n-k-1} and K^*_{k+1}, and therefore $D \in \mathcal{H}_{n,k}$, which proves the lemma. □

Lemma 6.1.5. *Let D be a digraph with $|D| = n \geq \frac{5}{2}k + 6$ and $||D|| \geq f(n,k)$. Then a symmetric path of order $n-k-1$ is contained in D.*

Proof. By the already proved part (1) of Theorem 6.1.1, there is a symmetric cycle C^*_{n-k-2} contained in D.

Suppose that the digraph D does not contain any symmetric path P^*_{n-k-1}. Consequently, it is impossible to extend C^*_{n-k-2} to a path P^*_{n-k-1}. Thus, the vertices of C^*_{n-k-2} can be joined with the vertices out of the cycle by at most $(n-k-2)(k+2)$ arcs. Hence

$$||D|| \leq (n-k-3)(n-k-2) + (n-k-2)(k+2) + (k+1)(k+2),$$

contradicting $||D|| \geq h(n,k)$. □

We now continue the proof of Case 2.

Combining Claims 1 and 2 with Lemmas 6.1.2, 6.1.4, 6.1.5, one can easily see that to conclude the proof it is sufficient to consider two cases of $n = 3k+4$ and $n = 3k+5$. In both cases we can make the following observations.

- Without loss of generality, we can assume from now on that $C^*_{n-k-2} \subset D$, $P^*_{n-k-1} \subset D$, $P^*_{n-k} \not\subset D$ and $V(C^*_{n-k-2}) \subset V(P^*_{n-k-1})$.

- If there are two vertices x and y not in the cycle C^*_{n-k-2} joined by a symmetric arc, then there is a cycle $C^{*'}_{n-k}$ contained in D.
  Indeed, then either D contains $C^{*'}_{n-k}$, or $a_D(x, x_i) + a_D(y, x_{i+1}) \leq 2$ for $i = 1, 2, \ldots, n-k-2$, where x_i and x_{i+1} are consecutive vertices of C^*_{n-k-2} (reducing indices modulo $n-k-2$). We deduce that $d_D(x) + d_D(y) \leq 2(n-k-2) + 4(k+1) = 2n + 2k$. But $d_D(x) + d_D(y) \geq 2(n+k)$, so $d_D(x) = d_D(y) = n+k$ and the above inequality was indeed an equality. Hence x and y are joined by symmetric arcs with all vertices not included in the cycle C^*_{n-k-2}. By the same argument, every vertex outside C^*_{n-k-2} has degree $n+k$. It follows:

  $$||D|| \leq (n-k-2)(n-k-3) + (n-k-2)(k+2) + (k+2)(k+1).$$

  An easy computation shows that it contradicts the assumption: $||D|| \geq f(n,k)$ for $n \geq 2k+4$.

  Hence, from now on we assume that no two vertices x and y out of the cycle C^*_{n-k-2} are adjacent by a symmetric arc.

- $d_D(v) \leq \frac{3}{2}n - \frac{1}{2}k - 2$ for every $v \in V(D) \setminus V(P^*_{n-k-1})$.

  Let $P^*_{n-k-1} = x_1x_2\ldots x_{n-k-1}$. Then $a_D(x_i, v) + a_D(x_{i+1}, v) \leq 3$ for $i = 1, 2, \ldots, n-k-2$, and $a_D(x_1, v) + a_D(x_{n-k-1}, v) \leq 2$, since $P^*_{n-k} \not\subset D$. It follows that $d_D(v) \leq \frac{3}{2}(n-k-2) + 1 + k = \frac{3}{2}n - \frac{1}{2}k - 2$.

Subcase 2.1. $n = 3k + 4$

Then we have $d_D(v) \geq n + k = 4k + 4$ and $d_D(v) \leq \frac{3}{2}n - \frac{1}{2}k - 2 = 4k + 4$, so $d_D(v) = 4k + 4$ for all vertices v out of $P^*_{n-k-1} = P^*_{2k+3} = x_1x_2 \ldots x_{2k+3}$. This condition gives us a special structure of the digraph D: the vertices of $V(D) \setminus \{x_1, x_2, \ldots, x_{2k+3}\}$ are joined with the vertices of P^*_{2k+3} with odd suffices by antisymmetric arcs, and by symmetric arcs with the vertices $x_2, x_4, \ldots, x_{2k+2}$. It is crucial to see that the vertices with odd suffices, for instance x_{2i+1} and x_{2j+1} $(0 \leq i < j \leq k+1)$, cannot be joined by symmetric arcs, since otherwise it is easy to construct a path P^*_{2k+4} in D:

$x_1x_2 ... x_{2i+1}x_{2j+1}x_{2j}x_{2j-1} ... x_{2i+2}vx_{2j+2}x_{2j+3} ... x_{2k+3} \subset D,\ 0 \leq i < j \leq k,$

$x_1x_2 \ldots x_{2i}vx_{2k+2}x_{2k+1} \ldots x_{2i+1}x_{2k+3} \subset D$ for $1 \leq i < j = k+1$,

and $vx_2x_3 \ldots x_{2k+3}x_1 \subset D$ for $i = 0$, $j = k+1$,

where v is any vertex of $V(D) \setminus \{x_1, x_2, \ldots, x_{2k+3}\}$.

In D, we have $k+1$ vertices of $V(D) \setminus V(P^*_{2k+3})$ which induce a tournament $T^{\rightleftharpoons}_{k+1}$. These vertices are joined with $U_{k+2} = \{x_1, x_3, \ldots, x_{2k+3}\}$ by $(k+1)(k+2)$ antisymmetric arcs, and with $W_{k+1} = \{x_2, x_4, \ldots, x_{2k+2}\}$ by $2(k+1)(k+1)$ symmetric arcs. The subdigraph induced in D by the set of vertices U_{k+2} has at most $\frac{(k+1)(k+2)}{2}$ arcs, and that induced by W_{k+1} has at most $k(k+1)$ arcs. Of course, there are at most $2(k+1)(k+2)$ arcs between W_{k+1} and U_{k+2}. Therefore

$$||D|| \leq \frac{k(k+1)}{2} + (k+1)(k+2) + 2(k+1)(k+1) +$$

$$+ \tfrac{(k+1)(k+2)}{2} + k(k+1) + 2(k+1)(k+2) = (k+1)(7k+9).$$

But $h(n, k) = h(3k+4, k) = (k+1)(7k+9)$. It means that the digraph D consists of two tournaments $T^{\rightleftharpoons}_{k+1}$ and $T^{\rightleftharpoons}_{k+2}$ joined by $(k+1)(k+2)$ antisymmetric arcs and a complete symmetric digraph K^*_{k+1} which is joined with $T^{\rightleftharpoons}_{k+1}$ and $T^{\rightleftharpoons}_{k+2}$ by $2(k+1)(2k+3)$ symmetric arcs. Hence the digraph D is the exception from the assertion *(3b)*.

Subcase 2.2. $n = 3k + 5$

Then $d_D(v) \geq n + k = 4k + 5$ and $d_D(v) \leq \frac{3}{2}n - \frac{1}{2}k - 2 = 4k + \frac{11}{2}$, so $d_D(v) = 4k + 5$ for every vertex v out of $P^*_{n-k-1} = P^*_{2k+4} = x_1x_2 \ldots x_{2k+4}$. It is easy to see that for every $u \in V(P^*_{2k+4})$ and $v \in V(D) \setminus V(P^*_{2k+4})$ we have $a_D(u, v) \geq 1$. Otherwise, there is in D a symmetric path of order $2k + 5 = n - k$. For every vertex $v \in V(D) \setminus V(P^*_{2k+4})$, we can partition the set $V(P^*_{2k+4})$ into two subsets, A_v and B_v say, such that every vertex of A_v is adjacent to v by an antisymmetric arc while every vertex of B_v is joined with v by a symmetric arc. Of course, we have $x_1, x_{2k+4} \in A_v$ and there are no consecutive vertices in P^*_{2k+4} that both belong to B_v. Clearly, we have $|A_v| = k+3$ and $|B_v| = k+1$. Moreover, there is exactly one $i_0 \in \langle 0, k+1 \rangle$ such that $x_{2i_0+1}, x_{2i_0+2} \in A_v$. Then

$$A_v = \{x_1, x_3, \ldots, x_{2i_0-1}, x_{2i_0+1}, x_{2i_0+2}, x_{2i_0+4}, \ldots, x_{2k+4}\},$$

$$\text{and} \quad B_v = \{\mathrm{x}_2, \mathrm{x}_4, \ldots, \mathrm{x}_{2\mathrm{i}_0}, \mathrm{x}_{2\mathrm{i}_0+3}, \mathrm{x}_{2\mathrm{i}_0+5}, \ldots, \mathrm{x}_{2\mathrm{k}+3}\}.$$

One can check that, since $P^*_{n-k} \not\subset D$, the only symmetric arc between two vertices of A_v is that joining x_{2i_0+1} and x_{2i_0+2}. These facts have the following consequences:

- the subdigraph induced by the vertices of $V(D) \setminus V(P^*_{2k+4})$ is a tournament $T^{\rightleftharpoons}_{k+1}$,
- there are at most $\frac{(k+2)(k+3)}{2} + 1$ arcs between the elements of A_v and at most $k(k+1)$ arcs in $D(B_v)$,
- A_v and B_v are joined by at most $2(k+1)(k+3)$ arcs,
- every vertex v of the $T^{\rightleftharpoons}_{k+1}$ (that is $D - V(P^*_{2k+4})$) is joined with the vertices of $V(P^*_{2k+4})$ by exactly $3k+5$ arcs (independently of the choice of v), since $d_D(v) = 4k+5$.

Thus

$$||D|| \leq \frac{k(k+1)}{2} + \frac{(k+2)(k+3)}{2} + 1 + k(k+1) + 2(k+1)(k+3) +$$

$$+ (k+1)(3k+5) = 7k^2 + 20k + 15,$$

contradicting the assumption $||D|| \geq h(n,k) = h(3k+5,k) = = 7k^2 + 21k + 16$.

□

As a consequence of Theorem 6.1.1 we get the following result, which is a generalization of Corollary 4.3.15, quoted at the beginning of this section. It is a solution of Problem 3.

Corollary 6.1.6 (LA and Wojda [5]). *Let D be a digraph. Suppose that $|D| = n$, $n \geq 7$, $n \geq \frac{5}{2}k + 6$, and $||D|| \geq h(n,k) = (n-k-1)(n-1) + k(k+1)$. Then:*

(1) D contains every orientation of a cycle of length $n-k$ except the strong one in case $D = K^*_{n-k-1} \to K^*_{k+1}$ or $D = K^*_{k+1} \to K^*_{n-k-1}$,

(2) D contains every orientation of C_p for all $3 \leq p \leq n-k-1$.

Proof. Part (2) is obvious, since by Theorem 6.1.1 the digraph D contains almost symmetric cycles of all lengths up to $n-k-1$. It remains to prove the first part of the corollary.

Theorem 6.1.1 asserts that either D contains an almost symmetric cycle $C^{*'}_{n-k}$, and hence every orientation of a cycle of length $n-k$, or D is an exceptional digraph. Therefore we need to consider the following three cases.

Case (a) D is a digraph of the family $\mathcal{H}_{n,k}$.

It is easy to see that every cycle of length $n-k$ is contained in D except the strong one when either $D = K^*_{n-k-1} \to K^*_{k+1}$ or $D = K^*_{k+1} \to K^*_{n-k-1}$.

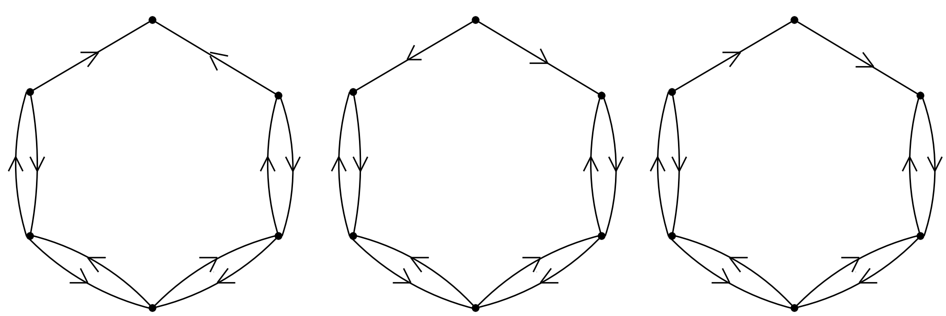
Case (b) $n = 3k+4$ and $D = T^{\rightleftharpoons}_{2k+3} * K^*_{k+1}$.

We have to find all the orientations of cycles of length $n-k = 2k+4$ in D. It is sufficient to show that D contains Z_{2k+4} and at least one of X_{2k+4} or Y_{2k+4}, where X_m, Y_m, Z_m denote the digraphs obtained from C^*_m by deleting two consecutive arcs, as shown in Figure 2. But this is evident since, for all vertices $x, y \in V(T^{\rightleftharpoons}_{2k+3})$, $x \neq y$, there is a symmetric path in D of order $2k+3$ starting at x and terminating at y.

Case (c) $n = 3k+2$ and $D = T^{\rightleftharpoons}_{2k+2} * K^*_{k}$.

The proof is similar to that of Case (b).

□



X_6 Y_6 Z_6

Figure 2

It is worth to mention here that the above corollary is a partial (for long cycles) generalization of a result by Häggkvist and Thomassen from 1976. They found, for every $3 \leq p \leq n$, the minimal size of a digraph D on n vertices which guarantees the existence of the strong orientation of a cycle C_p in D.

Consider the following digraph D of order n, where $n = q(p-1) + r$ and $0 \leq r \leq p-1$: D consists of the union of $q+1$ complete digraphs $F_1, F_2, \ldots, F_{q+1}$ such that one of them has order r and the remaining ones have order $p-1$. Then add all arcs of the type $(x \to y)$, where $x \in V(F_i)$, $y \in V(F_j)$ with $i < j$. In this way we get the family of digraphs, denoted by $\mathcal{H}_n$. Every digraph of $\mathcal{H}_n$ has

$$h_1(n,p) = \frac{1}{2}n(n-1) + \frac{1}{2}(n-r)(p-2) + \frac{1}{2}r(r-1)$$

arcs, and contains no strong cycle of length p or more.

Theorem 6.1.7 (Häggkvist and Thomassen [19, 41]). *A digraph of order n with more than $h_1(n,p)$ arcs contains a strong cycle of length p. If $p \geq 4$, then the only digraphs of size $h_1(n,p)$ which do not contain $C_p^{\to}$ are the digraphs of the family $\mathcal{H}_n$.*

Observe that for $p > \frac{n}{2} + 1$ the family $\mathcal{H}_n$ consists of the digraph $K^*_{p-1} \to K^*_{n-p+1}$ and its converse. Take $p = n-k$ to see that Corollary 6.1.6 generalizes Theorem 6.1.7: if D is a digraph with n vertices and at least $h_1(n,p)$ arcs, then, if only $p \geq \frac{3}{5}n + \frac{12}{5}$, D contains every orientation of a cycle C_p unless $D \in \mathcal{H}_n$. The analogous generalization for shorter cycles seems to be hard.

6.2 Cycles in bipartite digraphs

Throughout this section, $D = (X, Y; A)$ will denote a bipartite digraph with the vertex classes X and Y and the set of arcs A.
Let us recall that $\overline{D^{bip}}$ denote the complement of D to the complete bipartite digraph $K^*_{|X|,|Y|}$.

In this section, we are interested in solving a similar problem to Problem 3 in bipartite digraphs.

Before presenting results we have to define a special family of bipartite digraphs. Let $\mathcal{D}_{m,n,k} = \{\ D = (X, Y; A) :\ |X| = m \leq n = |Y|,\ X = U \cup W$, $U = \{x \in X : d_D(x) = 2n\}$, $W = \{x \in X : d_D(x) = n\}$, $|W| = k+1$, and each vertex of W is joined with all the vertices of Y by antisymmetric arcs $\}$. We distinguish special digraphs $D^1_{m,n,k}$ and $D^2_{m,n,k}$ of the family $\mathcal{D}_{m,n,k}$: in $D^1_{m,n,k}$ all antisymmetric arcs between the vertices of the sets W and Y are oriented from W to Y, while in $D^2_{m,n,k}$ they are oriented from Y to W.

The following result is the bipartite counterpart of Theorem 4.3.9.

Theorem 6.2.1 (Chakroun, Manoussakis and Manoussakis [28]). *Let $D = (X, Y; A)$ be a bipartite digraph, where $|X| = |Y| = n \geq 3$, and $||D|| \geq 2n^2 - n$. Then D contains a strong hamiltonian cycle unless $D = D^1_{n,n,0}$ or $D = D^2_{n,n,0}$.*

Wojda and Woźniak improved this theorem. They got a result about other orientations of a hamiltonian cycle as a consequence of the following theorem about almost symmetric hamiltonian cycles in balanced bipartite digraphs.

Theorem 6.2.2 (Wojda and Woźniak [59]). *Let $D = (X, Y; A)$ be a bipartite digraph, where $|X| = |Y| = n$, and $||D|| \geq 2n^2 - 2n + 3$. Then D contains an almost symmetric hamiltonian cycle unless there is a vertex in D which is not incident to any symmetric arc of D.*

Corollary 6.2.3 (Wojda and Woźniak [59]). *Let $D = (X, Y; A)$ be a bipartite digraph, where $|X| = |Y| = n \geq 3$, and $||D|| \geq 2n^2 - n$. Then D contains every orientation of a cycle of any even length up to $2n$ unless $D = D^1_{n,n,0}$ or $D = D^2_{n,n,0}$ (in both cases D does not contain any strong hamiltonian cycle).*

We would like to generalize the above theorem to long cycles in general bipartite digraphs. We consider the following problem.

Problem 4. For a bipartite digraph $D = (X, Y; E)$ with $|X| = m \leq n = |Y|$, and an integer $k < \frac{m}{2}$, find the minimum number of arcs of D, $z(m, n, k)$, which guarantees the existence of every orientation of a cycle of length $2m - 2k$ in D.

We partially (for $m \geq \frac{1}{2}k^2 + \frac{5}{2}k + 6$) solve the above problem (see Corollary 6.2.5). Again, the solution is a consequence of the following theorem about symmetric and almost symmetric cycles in bipartite graphs.

Theorem 6.2.4 (LA [3]). *Let $D = (X, Y; A)$ be a bipartite digraph with $|X| = m \leq n = |Y|$, where $m \geq \frac{1}{2}k^2 + \frac{5}{2}k + 6$. If $||D|| \geq z(m, n, k) = 2mn - n(k + 1)$, then:*

(1) D contains all symmetric cycles C^*_{2p} for $2 \leq p \leq m - k - 1$,

(2) D contains an almost symmetric cycle $C^{*'}_{2m-2k}$ unless $D \in \mathcal{D}_{m,n,k}$.

Before presenting the proof of the above theorem we have to introduce a useful notion. Similarly as it was in the previous section, for a digraph $D = (X, Y; A)$, the *associated graph* $G(D)$ is defined to be the undirected bipartite graph with the same vertex set and the same vertex classes as D, and such that xy is an edge of $G(D)$ precisely when x and y are joined by a symmetric arc in D. We shall tacitly use an obvious fact that $||G(D)|| \geq n^2 - l$ for a balanced bipartite digraph D with $2n$ vertices, whenever $||\overline{D^{bip}}|| \leq l$.

In the course of the proof we will apply Theorem 5.2.1 to the graph $G(D)$. Therefore, let us recall that $g(m, n, k) = n(m - k - 1) + k + 1$

Proof of Theorem 6.2.4. The proof will be divided into two steps. We first show that Theorem 6.2.4 is true for $m = n$. In the second step we will consider the case when $m < n$.

Step 1. $m = n$

Part (1) of Theorem 6.2.4 is a simple corollary of Theorem 5.2.1.

Indeed, observe that for a balanced bipartite digraph D with $2n$ vertices and at least $z(n,n,k)$ arcs we have $||\overline{D^{bip}}|| \leq n(k+1)$. Hence, for the associated graph $G(D)$, $||G(D)|| \geq n^2 - n(k+1) = n^2 - nk - n$.
For $n \geq \frac{1}{2}k^2 + \frac{5}{2}k + 6$, have $n \geq \frac{1}{2}(k+1)^2 + \frac{3}{2}(k+1) + 4$, and $n^2 - nk - n \geq n(n-k-2) + k + 3 = g(n,n,k+1) + 1$, so can apply Theorem 5.2.1 to the graph $G(D)$. Thus $C_{2p} \subset G(D)$ for all $2 \leq p \leq n-k-1$, so that $C^*_{2p} \subset D$ for all $2 \leq p \leq n-k-1$.

For the proof of part (2) of Theorem 6.2.4, we will proceed by induction on k.

Set $k = 0$. We want to show that, if D is a balanced bipartite digraph with $|D| = 2n \geq 12$ and $||D|| \geq 2n^2 - n$, then there exists an almost symmetric hamiltonian cycle in D, unless $||D|| = 2n^2 - n$ and $D \in \mathcal{D}_{n,n,0}$.

Since $2n^2 - n \geq 2n^2 - 2n + 3$ for $n \geq 3$, then by Theorem 6.2.2, either D contains an almost symmetric cycle of length $2n - 2k$, or else there is a vertex in D which is not incident to any symmetric arc. In the latter case, the number of arcs of D implies that $D \in \mathcal{D}_{n,n,0}$.

Now assume the theorem holds for $k-1$, $k \geq 1$; we will prove that it holds also for k.
Let us start with the observation that $z(n,n,k) - z(n-1,n-1,k-1) = 3n - k - 2$. In the proof we shall consider two cases.

Case 1. There is a pair of vertices $x \in X$ and $y \in Y$ in the digraph D such that $d_D(x) + d_D(y) \leq 3n - k - 2$.

Then $D - \{x,y\}$ is a balanced bipartite digraph with $2n-2$ vertices and at least $z(n,n,k) - (3n-k-2) = z(n-1,n-1,k-1)$ edges. By the inductive hypothesis there is an almost symmetric cycle $C^{*'}_{2(n-1)-2(k-1)} = C^{*'}_{2n-2k}$ contained in $D - \{x,y\}$, hence also in D, unless $D - \{x,y\} \in \mathcal{D}_{n-1,n-1,k-1}$. In the latter case, $d_D(x) + d_D(y) = 3n - k - 2$ and x and y are not adjacent, because $||D - \{x,y\}|| = z(n-1,n-1,k-1)$. What's more, $X \cap V(D - \{x,y\}) = U \cup W$, where $U = \{v : d_{D-\{x,y\}}(v) = 2n-2\}$, $W = \{v : d_{D-\{x,y\}}(v) = n-1\}$, $|W| = k$, and each vertex of W is joined with all the vertices of $Y \setminus \{y\}$ by antisymmetric arcs.
It is easy to check that $C^{*'}_{2n-2k} \subset D$ if $d_D(x) \geq n$ (then at least one vertex of Y is joined with x by a symmetric arc). It thus remains to consider what happens when $d_D(x) \leq n-1$. Then $d_D(y) \geq 2n-k-1$, which implies that $a_D(y,U) \geq n-k$ and $a_D(y,W) \geq k+1$ if only $n > 2k$. It means that there are vertices: u in U and w in W, which are joined with the vertex y by symmetric arcs. In this case it is easy to find a cycle $C^{*'}_{2n-2k}$ contained in D passing through the vertices u, y, and w.

Case 2. For every pair of vertices $x \in X$ and $y \in Y$ in the digraph D, we have $d_D(x) + d_D(y) \geq 3n - k - 1$.

By the already proved part (1) of Theorem 6.2.4, there is a symmetric cycle $C^*_{2n-2k-2} = x_1y_1x_2y_2 \ldots x_{n-k-1}y_{n-k-1}x_1$ contained in D. We can partition the set of vertices of D into two subsets: vertices of the set $V(C^*_{2n-2k-2})$ and the remaining ones, which define a set denoted by V_{2k+2}.

Suppose that a cycle $C^{*'}_{2n-2k}$ is not contained in D. We need to consider several subcases, depending on the way the vertices of V_{2k+2} are connected.

Subcase 2.1. There is a pair of vertices $x \in X \cap V_{2k+2}$ and $y \in Y \cap V_{2k+2}$ which are joined by a symmetric arc.

Since $C^{*'}_{2n-2k} \not\subset D$, we have $a_D(x, y_i) + a_D(y, x_i) \leq 2$, $i = 1, 2, \ldots, n - k - 1$, for otherwise we could extend the cycle $C^*_{2n-2k-2}$ to an almost symmetric cycle of length $2n - 2k$ using connections between the vertices x_i, y, x, and y_i. It follows that $a_D(x, V(C^*_{2n-2k-2})) + a_D(y, V(C^*_{2n-2k-2})) \leq 2(n - k - 1)$. Hence $d_D(x) + d_D(y) \leq 2(n - k - 1) + 2(2k + 2)$. But, by assumption made at the beginning of Case 2, $d_D(x) + d_D(y) \geq 3n - k - 1$. Both inequalities imply $n \leq 3k + 3$, contrary to the assumption $n \geq \frac{1}{2}k^2 + \frac{5}{2}k + 6$.

Subcase 2.2. No two vertices of the set V_{2k+2} are adjacent.

Let $x \in X \cap V_{2k+2}$ and $y \in Y \cap V_{2k+2}$. Similarly as in the previous case, since $C^{*'}_{2n-2k} \not\subset D$, we have $a_D(x, y_i) + a_D(y, x_i) + a_D(x, y_{i+1}) + a_D(y, x_{i+1}) \leq 6$, $i = 1, 2, \ldots, n - k - 1$ (reducing indices modulo $n - k - 1$). If not, we could extend the cycle $C^*_{2n-2k-2}$ to an almost symmetric cycle of length $2n-2k$ using connections between the vertices x_i, y, x_{i+1}, y_i, x, and y_{i+1}. We deduce that $2d_D(x) + 2d_D(y) \leq 6(n - k - 1)$, contradicting $d_D(x) + d_D(y) \geq 3n - k - 1$.

Subcase 2.3. There exist adjacent vertices in V_{2k+2}, but they are joined only by antisymmetric arcs.

Let $x \in X \cap V_{2k+2}$ and $y \in Y \cap V_{2k+2}$ be adjacent vertices. Again, since $C^{*'}_{2n-2k} \not\subset D$, we have $a_D(x, y_i) + a_D(y, x_i) \leq 3$, $i = 1, 2, \ldots, n - k - 1$, for otherwise we could extend the cycle $C^*_{2n-2k-2}$ to an almost symmetric cycle of length $2n - 2k$ using connections between the vertices x_i, y, x, and y_i. Hence $a_D(x, V(C^*_{2n-2k-2})) + a_D(y, V(C^*_{2n-2k-2})) \leq 3(n - k - 1)$. It implies that $d_D(x) + d_D(y) \leq 3(n - k - 1) + 2(k + 1)$, because x and y can be joined with other vertices of the set V_{2k+2} only by antisymmetric arcs. But $d_D(x) + d_D(y) \geq 3n - k - 1$, so $d_D(x) + d_D(y) = 3n - k - 1$ and the above inequality was, in fact, an equality, and we get that $d_{D(V_{2k+2})}(x) = d_{D(V_{2k+2})}(y) = k + 1$. Applying the same argument to every pair of adjacent vertices of the set V_{2k+2}, we obtain that the induced digraph $D(V_{2k+2})$ is a balanced bipartite tournament $T^{\rightleftharpoons}_{k+1,k+1}$. What's more, for every $x \in X \cap V_{2k+2}$ and $y \in Y \cap V_{2k+2}$, $a_D(x, V(C^*_{2n-2k-2})) + a_D(y, V(C^*_{2n-2k-2})) = 3(n-k-1)$. Therefore $||D|| \leq 2(n-k-1)^2 + 3(n-k-1)(k+1) + (k+1)^2 = 2n^2 - nk - n = z(n,n,k)$. It means that $||D|| = z(n,n,k)$ and vertices of $V(C^*_{2n-2k-2})$ induce a complete bipartite symmetric digraph $K^*_{n-k-1,n-k-1}$. There are exactly $3(n-k-1)(k+1)$ arcs between vertices of the $T^{\rightleftharpoons}_{k+1,k+1}$ and the $K^*_{n-k-1,n-k-1}$, and hence some of these arcs are symmetric. Without loss of generality we can assume that there is a vertex $v \in Y \cap V(T^{\rightleftharpoons}_{k+1,k+1})$ joined by a symmetric arc with $x_i \in X \cap V(K^*_{n-k-1,n-k-1})$ for some $i \in \{1, \ldots, n-k-1\}$. Since $C^{*'}_{2n-2k} \not\subset D$, there are no symmetric arcs between vertices of the sets $X \cap V(T_{k+1,k+1})$ and $Y \cap V(K^*_{n-k-1,n-k-1})$. Thus

$$a_D(X \cap V(T^{\rightleftharpoons}_{k+1,k+1}), Y \cap V(K^*_{n-k-1,n-k-1})) \leq (k+1)(n-k-1).$$

Of course,

$$a_D(X \cap V(K^*_{n-k-1,n-k-1}), Y \cap V(T^{\rightleftharpoons}_{k+1,k+1})) \leq 2(k+1)(n-k-1).$$

The number of arcs in D implies that the above inequalities are, in fact, equalities, and $D \in \mathcal{D}_{n,n,k}$.

Step 2. $m < n$

First order the vertices of $Y = \{y^1, \ldots, y^n\}$ so that $d_D(y^1) \leq \ldots \leq d_D(y^n)$. Put $Y' = \{y^1, \ldots, y^{n-m}\}$.

We shall consider two cases depending on the degree of the vertex y^{n-m}.

Case 1. $d_D(y^{n-m}) \geq 2m - k$

Then $D - Y'$ is a balanced bipartite digraph on $2m$ vertices such that $||D - Y'|| \geq m(2m-k) > z(m,m,k)$ since $m > 0$. From what has already been proved in Step 1, we conclude that $D - Y' \supset C^*_{2p}$ for all $2 \leq p \leq m-k-1$ and $D - Y' \supset C^{*'}_{2m-2k}$, which completes the proof in this case.

Case 2. $d_D(y^{n-m}) \leq 2m - k - 1$

Now we have $||D - Y'|| \geq z(m,n,k) - (n-m)(2m-k-1) = 2m^2 - mk - m = z(m,m,k)$, which again implies $D - Y' \supset C^*_{2p}$ for all $2 \leq p \leq m-k-1$, and also $D - Y' \supset C^{*'}_{2m-2k}$ unless $||D - Y'|| = z(m,m,k)$ and $D - Y' \in \mathcal{D}_{m,m,k}$. Therefore it remains to consider the case when $X = U \cup W$, where $U = \{v : d_{D-Y'}(v) = 2m\}$, $W = \{v : d_{D-Y'}(v) = m\}$, $|W| = k+1$, and each vertex of W is joined with all the vertices of $Y \setminus Y'$ by antisymmetric arcs. Then every vertex of Y' has degree in D equal $2m-k-1$ since $||D - Y'|| = z(m,m,k)$.

Suppose now that any almost symmetric cycle of length $2m-2k$ is not contained in D. Consequently, no vertex of Y' can be joined with both a vertex of U and a vertex of W by symmetric arcs, for otherwise we could simply construct a cycle $C^{*'}_{2m-2k}$ in D. Hence the assumption: $d_D(y^i) = 2m-k-1$ for all $1 \leq i \leq n-m$, implies (for $m > 2k+2$) that every vertex of Y' is joined with all vertices of U by symmetric arcs and with all vertices of W by antisymmetric arcs. It follows that $D \in \mathcal{D}_{m,n,k}$, which completes the proof.

□

Theorem 6.2.4 asserts that either D contains an almost symmetric cycle $C^{*'}_{2m-2k}$, and hence every orientation of a cycle of length $2m-2k$, or D is an exceptional digraph of the family $\mathcal{D}_{m,n,k}$. In the latter case, it is easy to see that every orientation of a cycle of length $2m-2k$ is contained in D except the strong one in case $D = D^1_{m,n,k}$ or $D = D^2_{m,n,k}$. Hence we have the following corollary (observe that for $m = n \geq 6$ and $k = 0$ it is exactly Corollary 6.2.3), which is the bipartite counterpart of Corollary 6.1.6, and simultaneously the directed counterpart of Theorem 5.2.1.

Corollary 6.2.5 (LA [3]). *Let $D = (X, Y; A)$ be a bipartite digraph with $|X| = m \leq n = |Y|$, where $m \geq \frac{1}{2}k^2 + \frac{5}{2}k + 6$. Suppose that*

$$||D|| \geq z(m, n, k) = 2mn - n(k+1).$$

Then:

(1) D contains every orientation of a cycle of length $2m-2k$ except the strong one in case $D = D^1_{m,n,k}$ or $D = D^2_{m,n,k}$,

(2) D contains every orientation of C_{2p} for all $2 \leq p \leq m-k-1$.

□

We expect that the assumption $m \geq \frac{1}{2}k^2 + \frac{5}{2}k + 6$ in Theorem 6.2.4 and in Corollary 6.2.5 may be relaxed; that is, that m can be assumed to be linear compared to k. The present quadratic bound on m is simply a consequence of the assumptions of Theorem 5.2.1, which we use in the proof of part (1) of Theorem 6.2.4.
The essential assumption on m in the proof of part (2) of Theorem 6.2.4 is $m > 3k+3$ (see Subcase 2.1. in Step 1). Thus the condition on m would be weakened automatically if only we could improve Theorem 5.2.1, for example by proving Conjecture 5.2.4.

Finally, we show that the bound on m cannot be less than $2k+2$ by constructing the special digraph D_1 as follows.

Example 6.2.6. Let $D_1 = (X, Y; E)$ be a bipartite digraph, with vertex classes of the form $X = Q \cup R$, $Y = S \cup T$, where $|Q| = |T| = m-k-1$, $|R| =$

$k+1$, $|S| = n-m+k+1$, $m \leq n$. Let $N^+_{D_1}(x) = Y$, $N^-_{D_1}(x) = S$ for all $x \in Q$, and $N^+_{D_1}(x) = T$, $N^-_{D_1}(x) = Y$ for all $x \in R$. Then $\|D_1\| > z(m,n,k) = 2mn - n(k+1)$ for $k+2 \leq m \leq 2k+1$, yet D_1 contains no strong cycle of length greater than $2m-2k-2$.

We formulate the following conjecture.

Conjecture 6.2.7. *Let $D = (X, Y; A)$ be a bipartite digraph with $|X| = m \leq n = |Y|$, where $m \geq 2k+2$. Suppose that $||D|| > z(m,n,k) = 2mn - n(k+1)$. Then D contains every orientation of a cycle C_{2p} for all $2 \leq p \leq m-k$.*

Example 6.2.6 shows also that there is no chance to find a simple analogue of the result due to Häggkvist and Thomassen (Theorem 6.1.7) in bipartite digraphs. One has to consider at least two cases: long and short cycles. Short cycles probably require a different approach than the presented above.

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