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The Resultants and Discriminants of Real Polynomials and Quasipolynomials

1. Introduction

We shall start with the notion of the resultant

Theorem 1

Let $M(s)$ and $L(s)$ be polynomials with real coefficients

$$\left. \begin{aligned} M(s) &= a_0 s^n + a_1 s^{n-1} + \dots + a_n \quad (a_0 \neq 0) \\ L(s) &= b_0 s^m + b_1 s^{m-1} + \dots + b_m \quad (b_0 \neq 0) \end{aligned} \right\} \quad (1)$$

The determinant (2) of order $m + n$ is said to be the Sylvester determinant or resultant.

$$R(M, L) = \left| \begin{array}{cccccccccc} a_0 & a_1 & \cdots & a_{n-1} & a_n & 0 & \cdots & 0 & 0 \\ 0 & a_0 & \cdots & a_{n-2} & a_{n-1} & a_n & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & & & & \cdots & \vdots & \\ 0 & 0 & \cdots & a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n \\ b_0 & b_1 & \cdots & b_{m-1} & b_m & 0 & \cdots & 0 & 0 \\ 0 & b_0 & \cdots & b_{m-2} & b_{m-1} & b_m & \cdots & 0 & 0 \\ & \vdots & \cdots & & & & \cdots & & \\ 0 & 0 & \cdots & b_0 & b_1 & b_2 & \cdots & b_{m-1} & b_m \end{array} \right| \quad \left. \begin{array}{l} \left. \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right\} m \text{ rows} \\ \left. \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right\} n \text{ rows} \end{array} \right\} \quad (2)$$

Relationships between the zeroing of the resultant and the existence of common zeros of the polynomials M and L are the most important issues considered in the paper.

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2. Discriminants of the polynomials

A special case of the resultant is the discriminant.

Consider the polynomial

$$M(s) = a_0 s^n + a_1 s^{n-1} + \dots + a_n \quad (a_0 \neq 0) \quad (3)$$

And its derivative with respect to the variable „s”

$$\frac{dM(s)}{ds}(s) = na_0 s^{n-1} + (n-1)a_1 s^{n-2} + \dots + a_{n-1} \quad (4)$$

The discriminant of these polynomials for $n > 2$ is:

$$D[M(s)] = (-1)^{\frac{n(n-1)}{2}} \frac{1}{a_0} R \left[M(s), \frac{dM(s)}{ds}(s) \right] \quad (5)$$

We take also into account that polynomial (3) can be written in the form

$$M(s) = a_0(s-s_1)(s-s_2) \dots (s-s_n) \quad (6)$$

where s_i – are the roots of $M(s)$.

Using (3), (4) and (5) we can write

$$D(M) = (-1)^{\frac{n(n-1)}{2}} \left\{ \begin{array}{c} \left| \begin{array}{cccccc} 1 & a_1 & \cdots & a_n & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_n & \cdots & 0 \end{array} \right| \\ \hline \left| \begin{array}{cccccc} & & a_0 & \cdots & & a_n \\ n & (n-1) & a_1 & \cdots & a_{n-1} & \cdots & 0 \\ & & & & na_0 & \cdots & a_{n-1} \end{array} \right| \end{array} \right\} \quad (7)$$

Example

Let be $M(s) = a_0 s^3 + a_1 s^2 + a_2 s + a_3$.

Then

$$D(M) = (-1)^{\frac{3(3-1)}{2}} \left| \begin{array}{ccccc} 1 & a_1 & a_2 & a_3 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ 3 & 2a_1 & a_2 & 0 & 0 \\ 0 & 3a_0 & 2a_1 & a_2 & 0 \\ 0 & 0 & 3a_0 & 2a_1 & a_2 \end{array} \right| = a_1^2 a_2^2 - 4a_1 a_3^3 - 4a_0 a_2^3 + 18a_0 a_1 a_2 a_3 - 27a_0^2 a_3^2.$$

For the product of the discriminants we have a formula [1]

$$D[M, L] = D[M] D[L] [R(M, L)]^2 \quad (8)$$

The following generalization of (8) is possible

$$D \left[\prod_{k=1}^n M_k \right] = \prod_{k=1}^n D(M_k) \cdot \prod_{i>j}^{1,n} [R(M_i M_j)]^2 \quad (9)$$

If we put $M_k(s) = s - s_k$,

$$\text{then } D[M_k] = 1 \text{ and } R[M_i, M_j] = \begin{vmatrix} 1 & -s_i \\ 1 & -s_j \end{vmatrix} = s_i - s_j.$$

Using (9) we obtain that

$$D \left[\prod_{k=1}^n (s - s_k) \right] = \prod_{i>j}^{1,n} (s_i - s_j)^2 \quad (10)$$

Taking into account the relation (6) we have

$$D = \prod_{i>j}^{1,n} (s_i - s_j)^2 = (-1)^{\frac{n(n-1)}{2}} \left| \begin{array}{ccccccc} 1 & a_1 & \cdots & a_n & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_n & \cdots & 0 \\ \hline & & a_0 & \cdots & & & a_n \\ n & (n-1)a_1 & \cdots & a_{n-1} & \cdots & 0 & \\ \hline & & & na_0 & \cdots & a_{n-1} & \end{array} \right| \begin{matrix} \left. \vphantom{\begin{matrix} 1 \\ 0 \\ \hline \\ n \end{matrix}} \right\} n-1 \\ \left. \vphantom{\begin{matrix} a_n \\ a_{n-1} \\ a_{n-1} \end{matrix}} \right\} n \end{matrix} \quad (11)$$

We know also that Vandermond's determinant is equal

$$V = \left| \begin{array}{cccc} 1 & 1 & \cdots & 1 \\ s_1 & s_2 & \cdots & s_n \\ s_1^2 & s_2^2 & \cdots & s_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ s_1^{n-1} & s_2^{n-1} & \cdots & s_n^{n-1} \end{array} \right| = \prod_{i>j}^{1,n} (s_i - s_j) \begin{matrix} \left. \vphantom{\begin{matrix} 1 \\ s_1 \\ s_1^2 \end{matrix}} \right\} n-1 \\ \left. \vphantom{\begin{matrix} s_n^2 \\ \vdots \\ s_n^{n-1} \end{matrix}} \right\} n \end{matrix} \quad (12)$$

From (11) and (12) we obtain the important relation

$$D = V^2 \quad (13)$$

In control theory we meet also other resultants [2].

Let us consider the characteristic equation

$$M(s) = 0 \quad (14)$$

We put $s = \alpha + j\omega$ in (14) after reordering we obtain two equations for the real and imaginary parts of the equation (14)

$$\begin{cases} P(\omega) = \operatorname{Re} M(\alpha + j\omega) \\ Q(\omega) = \operatorname{Im} M(\alpha + j\omega) \end{cases} \quad (15)$$

where the coefficients of the polynomials $P(\omega)$ and $Q(\omega)$ depend on „ α ”

$$P(\omega) = \sum_{0 \leq 2i \leq n} (-1)^i A_{2i}(\alpha) \omega^{2i} \quad (16)$$

$$Q(\omega) = \sum_{1 \leq 2i+1 \leq n} (-1)^i A_{2i}(\alpha) \omega^{2i+1} \quad (17)$$

We obtain

Theorem 2

If

$$M(s) = a_0 \prod_{i=0}^n (s - s_i) \quad a_0 \neq 0,$$

then

$$R[P, Q] = a_0^{2n} (-1)^{\frac{n(n+1)}{2}} 2^{n(n-1)} \prod_{i,k=1}^n \left(\alpha - \frac{s_i + s_k}{2} \right) \quad (18)$$

It is possible [2] to obtain simpler formulae

$$R[P, Q] = a_0^2 M(s) R_1^2(\alpha) \quad (19)$$

where

$$R_1^2(\alpha) = \left[a_0^{n-1} 2^{\frac{n(n-1)}{2}} \prod_{\substack{i,k=1 \\ i < k}}^n (\alpha - \alpha_{ik}) \right]^2 \quad (20)$$

and $\alpha_{ik} = \frac{s_i + s_k}{2}$.

It is worth noting that the free term of the polynomial $R_1(\alpha)$ is identical with the Hurwitz determinant Δ_{n-1} . From this we conclude that

$$\Delta_{n-1} = a_0^{n-1} (-1)^{\frac{n(n-1)}{2}} \prod_{\substack{i,k=1 \\ i < k}}^n (s_i + s_k) \quad (21)$$

which represents the well known Orlando formula.

Following this way with polynomials $U(\alpha)$ and $V(\alpha)$ having coefficients dependent on „ ω “, where

$$\left. \begin{aligned} U(\alpha) &= \operatorname{Re} M(j\omega + \alpha) \\ V(\alpha) &= \operatorname{Im} M(j\omega + \alpha) \end{aligned} \right\} \quad (22)$$

we have

$$U(\alpha) = \frac{1}{2} \left[(B_0(\omega) + C_0(\omega)) + (B_1(\omega) + C_1(\omega)\alpha) + \dots + (B_n(\omega) + C_n(\omega)\alpha^n) \right] \quad (23)$$

$$V(\alpha) = \frac{1}{2j} \left[(B_0(\omega) + C_0(\omega)) + (B_1(\omega) - C_1(\omega)\alpha) + \dots + (B_n(\omega) - C_n(\omega)\alpha^n) \right] \quad (24)$$

where

$$\left. \begin{aligned} B_k(\omega) &= \frac{1}{k!} M_{(j\omega)}^{(k)} \\ C(\omega) &= \frac{1}{k!} M_{(j\omega)}^{(k)} \end{aligned} \right\} \quad k = 0, 1, \dots, n \quad (25)$$

represent (k) -th derivatives with respect to (ω) .

We can state

Theorem 3

If $M(s) = a_0 \prod_{i=0}^n (s - s_i)$,

then the resultant

$$R[U, V] = a_0^{2n} (-4)^{\frac{n(n-1)}{2}} \prod_{i,k=1}^n \left(\omega - \frac{s_i - s_k}{2j} \right) \quad (26)$$

It is also possible to obtain a simpler formula because

$$R[U, V] = a_0 (-1)^{\frac{n(n-1)}{2}} \omega^n R_2(\beta) \Big|_{\beta=\omega^2} \quad (27)$$

where

$$R_2[\beta] = a_0^{2n-1} 4^{\frac{n(n-1)}{2}} \prod_{\substack{i,k=1 \\ i < k}}^n (\beta - \beta_{ik}) \quad (28)$$

$$\text{and } \beta_{ik} = - \left[\frac{s_i - s_k}{2} \right]^2.$$

Finally we have

$$R_2[U, V] = a_0^{2n} (-4)^{\frac{n(n-1)}{2}} \omega^n \prod_{\substack{i,k=1 \\ i < k}}^n \left[\omega^2 + \left(\frac{s_i - s_k}{2} \right)^2 \right] \quad (29)$$

The free term $R_2(\beta)$ is identical to Meeroov determinant Γ_{2n-1} which is given in the criterion of aperiodic stability.

On the basis of the theorem 3 we obtain an analogue of the Orlando formula

$$\Gamma_{2n-1} = a_0^{2n} \prod_{\substack{i,k=1 \\ i < k}}^n \left(\frac{s_i - s_k}{2} \right)^2 \quad (30)$$

which for $a_0 = 1$ is also connected with the well known Vandermonde determinant.

$$V_n^2 = \Gamma_{2n-1} = \left| \begin{array}{cccccc} 1 & a_1 & \cdots & a_n, 0 & \cdots & 0 \\ 0 & & \cdots & a_1 & \cdots & a_n \\ \cdots & & & & & \\ n(n-1)a_1 & \cdots & & a_{n-1} & \cdots & 0 \\ 0 & \cdots & 0 & n(n-1)a_1 & \cdots & a_{n-1} \end{array} \right| \begin{array}{l} \left. \vphantom{\begin{array}{c} 1 \\ 0 \\ \cdots \\ n(n-1)a_1 \\ 0 \end{array}} \right\} n-1 \\ \left. \vphantom{\begin{array}{c} a_n, 0 \\ a_1 \\ \cdots \\ a_{n-1} \\ n(n-1)a_1 \end{array}} \right\} n \end{array} \quad (31)$$

We conclude that the obtained resultants allow us to unify the study of different kinds of stability of linear stationary dynamic systems. In particular the determinant Δ_{n-1} determined by (21) represents the limit of the oscillatory stability and the determinant Γ_{2n-1} (formula (30)) the limit of the aperiodic stability.

3. Resultants of quasipolynomials¹⁾

Let us consider the differential equation which describes a linear, stationary dynamic system with constant parameters $a_0, \dots, a_n$

$$a_0 \frac{d^n x(t)}{dt^n} + a_1 \frac{d^{n-1} x(t)}{dt^{n-1}} + \dots + a_{n-1} \frac{dx(t)}{dt} + a_n x(t) = 0 \quad (32)$$

We assume that not all initial conditions are equal to zero, ie

$$x^{(i-1)}(0) = c_i \neq 0 \text{ for some } i = 1, 2, \dots, n.$$

The solution of equation (32) takes the form

$$x(t) = \sum_{k=1}^n A_k e^{s_k t} \quad (33)$$

where s_k are different roots of the characteristic equation

$$a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n = 0 \quad (34)$$

We will also use the derivatives of $x(t)$

$$x^{(p)}(t) = \sum_{k=1}^n s_k^p A_k e^{s_k t} \quad p = 1, 2, \dots, n-1 \quad (35)$$

It is not possible to use discriminants directly, because we cannot eliminate „ n ” exponential terms from two equations (33) and (35) for $p = 1$.

The additional equations we obtain by an appropriate change of only one initial condition from c_i to c_i^* .

We adjust the value c_1^* in such a way that both $x(t_0)$ and $x^*(t_0)$ are equal to zero. All the values of other derivatives are the same.

Remark

In the case of the triple roots we can use de L'Hospital theorem.

¹⁾ quasipolynomials it is the expression of the form $a_0 s^{\alpha_n} + a_1 s^{\alpha_{n-1}} + \dots + a_{n-1} s^{\alpha_1} + a_n$ where $\{\alpha_r\}$ denote real or complex numbers

Theorem 4

It is proved in [4] that the resultant of the equations (33) and (35) for $p = 1, 2, \dots, n-1$ is equal to

$$R = \frac{\left(\prod_{i=1}^n s_i \right)^{n-2}}{\prod_{\substack{i,j=1 \\ i < j}} (s_i - s_j)} \left[(c_1 - c_1^*) \right]^{n-2} (c_2^2 - c_1^* c_3) \quad n \geq 2 \quad (36)$$

or in an equivalent form

$$R = \frac{a_n^{n-2}}{V} \left[(c_1 - c_1^*)^{n-2} (c_2^2 - c_1^* c_3) \right] \quad (37)$$

where V is Vandermonde's determinant whose square is equal according to (31), the determinant responsible for aperiodicity.

The resultant is equal to zero if

$$c_1^* = \frac{c_2^2}{c_3} \quad (38)$$

Example

Consider the differential equation of the 3rd order

$$\frac{d^3 x}{dt^3} + a_1 \frac{d^2 x}{dt^2} + \frac{dx}{dt} + a_3 x = 0$$

with initial conditions:

$$x(0) = c_1,$$

$$\dot{x}(0) = c_2,$$

$$\ddot{x}(0) = c_3.$$

The solution of this equation is $x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + A_3 e^{s_3 t}$ or in detail

$$\begin{aligned} x(t) = & \frac{c_3 - (s_2 + s_3)c_2 + s_2 s_3 c_1}{(s_1 - s_2)(s_1 - s_3)} e^{s_1 t} + \frac{c_3 - (s_1 + s_3)c_2 + s_1 s_3 c_1}{(s_2 - s_1)(s_2 - s_3)} e^{s_2 t} + \\ & + \frac{c_3 - (s_1 + s_2)c_2 + s_1 s_2 c_1}{(s_3 - s_1)(s_3 - s_2)} e^{s_3 t} \end{aligned}$$

where $s_i \neq s_j$ are the roots of the characteristic equation

$$s^3 + a_1 s^2 + a_2 s + a_3 = 0.$$

We can write that for the extremum at the point t_e it is required

$$\begin{aligned} A_1^* e^{s_1 t_e} + A_2^* e^{s_2 t_e} + A_3^* e^{s_3 t_e} &= 0, \\ s_1 A_1^* e^{s_1 t_e} + s_2 A_2^* e^{s_2 t_e} + s_3 A_3^* e^{s_3 t_e} &= 0, \\ s_1 A_1 e^{s_1 t_e} + s_2 A_2 e^{s_2 t_e} + s_3 A_3 e^{s_3 t_e} &= 0. \end{aligned}$$

Where A_1^*, A_2^*, A_3^* denote the coefficients with the new condition $x^*(0) = c_1^*$, and others remain the same.

The resultant of this equation is

$$R = \begin{vmatrix} A_1^* & A_2^* & A_3^* \\ s_1 A_1^* & s_2 A_2^* & s_3 A_3^* \\ s_1 A_1 & s_2 A_2 & s_3 A_3 \end{vmatrix} = 0.$$

After calculation we obtain

$$R = \frac{s_1 s_2 s_3}{(s_1 - s_2)(s_2 - s_3)(s_3 - s_1)} \left[(c_1^*)^2 c_3 - c_1^* (c_2^2 + c_1 c_3) + c_1 c_2^2 \right] = 0.$$

From this we obtain that $c_1^* = \frac{c_2^2}{c_3}$ or $c_1^* = c_1$.

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