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ADELIC ANALYSIS AND FUNCTIONAL ANALYSIS ON THE FINITE ADELE RING

Ilwoo Cho

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Abstract. In this paper, we study operator theory on the $*$-algebra $\mathcal{M}_{\mathcal{P}}$, consisting of all measurable functions on the finite Adele ring $A_{\mathbb{Q}}$, in extended free-probabilistic sense. Even though our $*$-algebra $\mathcal{M}_{\mathcal{P}}$ is commutative, our Adelic-analytic data and properties on $\mathcal{M}_{\mathcal{P}}$ are understood as certain free-probabilistic results under enlarged sense of (noncommutative) free probability theory (well-covering commutative cases). From our free-probabilistic model on $A_{\mathbb{Q}}$, we construct the suitable Hilbert-space representation, and study a C^*-algebra $M_{\mathcal{P}}$ generated by $\mathcal{M}_{\mathcal{P}}$ under representation. In particular, we focus on operator-theoretic properties of certain generating operators on $M_{\mathcal{P}}$.

Keywords: representations, C^*-algebras, p-adic number fields, the Adele ring, the finite Adele ring.

Mathematics Subject Classification: 05E15, 11G15, 11R47, 11R56, 46L10, 46L54, 47L30, 47L55.

1. INTRODUCTION

The main purposes of this paper are:

(i) to construct a free-probability model (under extended sense) of the $*$-*algebra* $\mathcal{M}_{\mathcal{P}}$ consisting of all *measurable functions* on the *finite Adele ring* $A_{\mathbb{Q}}$, implying number-theoretic information from the *Adelic analysis on* $A_{\mathbb{Q}}$,
(ii) to establish a suitable *Hilbert-space representation* of $\mathcal{M}_{\mathcal{P}}$, reflecting our free-distributional data from (i) on $\mathcal{M}_{\mathcal{P}}$,
(iii) to construct-and-study a C^*-*algebra* $M_{\mathcal{P}}$ generated by $\mathcal{M}_{\mathcal{P}}$ under our representation of (ii), and
(iv) to consider *free distributions* of the *generating operators of* $M_{\mathcal{P}}$ of (iii).

Our main results illustrate interesting connections between *primes* and *operators* via *free probability theory.*

1.1. PREVIEW

We have considered how *primes* (or *prime numbers*) act on *operator algebras*. The relations between primes and operator algebra theory have been studied in various different approaches. For instance, we studied how primes act on certain *von Neumann algebras* generated by p-adic and Adelic measure spaces (e.g., [2]). Meanwhile, in [1], primes are regarded as *linear functionals* acting on *arithmetic functions*. In such a case, one can understand arithmetic functions as *Krein-space operators* under certain *Krein-space representations*. Also, in [3, 4] and [7], we considered free-probabilistic structures on a *Hecke algebra* $\mathcal{H}(G_p)$ for *primes* p.

In [6], we considered certain *free random variables* in $*$-algebras $\mathcal{M}_p$ of all *measurable functions* on the p*-adic number fields* $\mathbb{Q}_p$ in terms of the p*-adic integrations* φ_p for all primes p. Under suitable *Hilbert-space representations* of $\mathcal{M}_p$, the corresponding C^**- algebras* M_p of $\mathcal{M}_p$ are constructed and C^*-probability on M_p is studied there. In particular, for all $j \in \mathbb{Z}$, we define C^*-probability spaces $\left(M_p, \varphi_j^p\right)$, where φ_j^p are kind of sectionized linear functionals implying the number-theoretic data on $\mathcal{M}_p$, in terms of φ_p. Moreover, from the system

$$\left\{\left(M_p, \varphi_j^p\right) : p \in \mathcal{P}, j \in \mathbb{Z}\right\},$$

of C^*-probability spaces, we establish-and-consider the *free product* C^**-probability space*,

$$(M_{\mathcal{P}}(\mathbb{Z}), \varphi) = \underset{p\in\mathcal{P},\ j\in\mathbb{Z}}{\star} \left(M_p,\ \varphi_j^p\right),$$

called the *Adelic* C^**-probability space*.

Independently, in [8], by using the free-probabilistic information from a single C^*-probability space $\left(M_p, \varphi_j^p\right)$ (also introduced as above in [6]), for arbitrarily fixed $p \in \mathcal{P}, j \in \mathbb{Z}$, we established a *weighted-semicircular element* in a certain *Banach* $*$*-probability space* generated by (M_p, φ_j^p), and realized that corresponding *semicircular elements* are well-determined. Motivated by [6], we extended the weighted-semicircularity, and semicircularity of [8] in the free product Banach $*$-probability space of Banach $*$-probability spaces of [8], over both primes and integers, in [5].

1.2. MOTIVATION AND DISCUSSION

Motivated by the main results of [5, 6] and [8], we here establish free-probabilistic models (under extended sense) started from the *finite Adele ring* $A_{\mathbb{Q}}$, to provide similar framworks of [5] and [8]. Even though our structures are based on the commutativity, and hence, they are not directly followed original noncommutative free probability theory, the proceeding processes, settings, and results are from free probability theory. Thus, we use concepts and terminology from free probability theory.

In this very paper, we will consider neither weighted-semicircularity nor semicircularity on our free-probabilistic structures, however, later, our main results would provide suitable tools and backgrounds for studying those *semicircular-like laws* and the *semicircular law*. Readers may realize from our main results that the *spectral properties* of operators induced by measurable functions on $A_{\mathbb{Q}}$, and those of *free reduced words* induced by measurable functions on $\mathbb{Q}_p$ (under free product over primes p, obtained in [6]) are very similar (under certain additional conditions). It guarantees that one may/can obtain weighted-semicircularity and semicircularity on our structures as in [5] and [8] in future.

1.3. OVERVIEW

In Section 2, we briefly introduce backgrounds and a motivation of our works.

Our free-probabilistic model on $\mathcal{M}_{\mathcal{P}}$ is established from Adelic calculus, and the free distributional data on $\mathcal{M}_{\mathcal{P}}$ are considered in Section 3. Then, in Section 4, we construct a suitable Hilbert-space representation of our free-probabilistic model of $\mathcal{M}_{\mathcal{P}}$, preserving the free-distributional data implying number-theoretic information. Under representation, the corresponding C^*-algebra $M_{\mathcal{P}}$ is constructed.

In Section 5, free probability on the C^*-algebra $M_{\mathcal{P}}$ is studied by putting a system of linear functionals dictated by the Adelic integration. In particular, free distributions of generating operators of $M_{\mathcal{P}}$ are considered by computing free moments of them.

In Sections 6, we further consider relations between our free-distributional data and Adelic-analytic information, by computing free distributions of generating operators of the free product C^*-algebras. Especially, we focus on the free distributions of certain generating opeartors, called $(+)$-*boundary operators*.

In Section 7, by constructing free product C^*-probability spaces (which are under usual sense of noncommutative free probability theory) from our system of C^*-probability spaces (which are under extended commutativity-depending sense) of Sections 5 and 6, we investigate the connections between Adelic analysis and our free-probabilistic structures. In the long run, we study noncommutative free probability theory induced by the Adelic analysis on the finite Adele ring $A_{\mathbb{Q}}$.

1.4. MAIN RESULTS

By applying free-probabilistic settings and terminology, we characterize the functional-analytic properties of the C^*-algebras induced by the finite Adele ring with free-probabilistic language. For instance, functional-analytic or spectral-theoretic information of certain operators in our C^*-algebras are determined by the forms of free moments, or joint free moments (see Sections 5, 6 and 7). Since our C^*-algebra $M_{\mathcal{P}}$ is commutative, the free-probabilistic model for $M_{\mathcal{P}}$ is non-traditional, but such a model in Sections 5 and 6 is perfectly fit to analize our main results (even though the freeness on it is trivial), moreover, this non-traditional approaches become traditional by constructing free product C^*-algebras in Section 7.

The constructions of our free-probabilistic models and corresponding free--distributional datas of operators are the main results of this paper. From these, one can see the connections between (number-theoretic) Adelic analysis, functional analysis, and (operator-theoretic) spectral theory via free probability.

2. PRELIMINARIES

In this section, we briefly mention about backgrounds of our proceeding works. See [9] and [10] (and cited papers therein) for number-theoretic motivations.

2.1. FREE PROBABILITY

Readers can check fundamental analytic-and-combinatorial *free probability* from [12] and [14] (and the cited papers therein). *Free probability* is understood as the *noncommutative* operator-algebraic version of classical *probability theory* and *statistics* (covering commutative cases). The classical *independence* is replaced by the *freeness*, by replacing *measures* to *linear functionals*. It has various applications not only in pure mathematics (e.g., [11]), but also in related scientific topics (for example, see [2, 3, 5] and [8]). In particular, we will use combinatorial approach of *Speicher* (e.g., [12]).

Especially, in the text, without introducing detailed definitions and combinatorial backgrounds, *free moments* and *free cumulants* of operators will be computed. Also, we use *free product of algebras in the sense of* [12] *and* [14], without detailed introduction.

2.2. p-ADIC CALCULUS ON $\mathbb{Q}_p$

In this section, we briefly review *p-adic calculus* on the $*$-algebras $\mathcal{M}_p$ of measurable functions on p-adic number fields $\mathbb{Q}_p$, for $p \in \mathcal{P}$. For more about p-adic analysis see e.g., [13].

For a fixed prime p, the p-adic number field $\mathbb{Q}_p$ is the maximal p-norm closure in the set $\mathbb{Q}$ of all *rational numbers*, where the *p-norm* $|\cdot|_p$ *on* $\mathbb{Q}$ is defined by

$$|x|_p = \left|a \cdot p^k\right|_p = \frac{1}{p^k},$$

whenever $x = ap^k$, for some $a \in \mathbb{Q}$, and $k \in \mathbb{Z}$. For instance,

$$\left|\frac{4}{3}\right|_2 = \left|\frac{1}{3} \cdot 2^2\right| = \frac{1}{2^2} = \frac{1}{4},$$
$$\left|\frac{4}{3}\right|_3 = \left|4 \cdot 3^{-1}\right| = \frac{1}{3^{-1}} = 3,$$

and

$$\left|\frac{4}{3}\right|_q = \left|\frac{4}{3} \cdot q^0\right| = \frac{1}{q^0} = 1, \text{ for all } q \in \mathcal{P} \setminus \{2, 3\}.$$

Every element z of $\mathbb{Q}_p$ is expressed by

$$z = \sum_{k=-N}^{\infty} a_k p^k, \text{ with } a_k \in \{0, 1, \ldots, p-1\}, \tag{2.1}$$

for some $N \in \mathbb{N}$. So, from the p-adic addition and the p-adic multiplication on (the elements formed by (2.1) under $\sum$ in) $\mathbb{Q}_p$, the set $\mathbb{Q}_p$ forms a ring algebraically (e.g., [13]).

Moreover, one can understand this *Banach ring* $\mathbb{Q}_p$ as a *measure space*,

$$\mathbb{Q}_p = \left(\mathbb{Q}_p,\ \sigma(\mathbb{Q}_p),\ \mu_p\right),$$

where $\sigma(\mathbb{Q}_p)$ is the *σ-algebra of* $\mathbb{Q}_p$ consisting of all μ_p*-measurable subsets*, where μ_p is a left-and-right additive-invariant *Haar measure on* $\mathbb{Q}_p$, satisfying

$$\mu_p(\mathbb{Z}_p) = 1,$$

where $\mathbb{Z}_p$ is the *unit disk*

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

of $\mathbb{Q}_p$, consisting of all *p-adic integers* x, having their forms

$$x = \sum_{k=0}^{\infty} a_k p^k \text{ with } a_k \in \{0, 1, \ldots, p-1\}.$$

Moreover, if we define

$$U_k = p^k \mathbb{Z}_p = \{p^k x \in \mathbb{Q}_p : x \in \mathbb{Z}_p\}, \tag{2.2}$$

for all $k \in \mathbb{Z}$, then these μ_p-measurable subsets U_k of (2.2) satisfy

$$\mathbb{Q}_p = \bigcup_{k \in \mathbb{Z}} U_k,$$

and

$$\mu_p\left(U_k\right) = \frac{1}{p^k} = \mu_p\left(x + U_k\right), \text{ for all } k \in \mathbb{Z},$$

and

$$\ldots \subset U_2 \subset U_1 \subset U_0 = \mathbb{Z}_p \subset U_1 \subset U_2 \subset \ldots \tag{2.3}$$

(e.g., [13]). In fact, the family $\{U_k\}_{k \in \mathbb{Z}}$ forms a *basis* of the Banach topology for $\mathbb{Q}_p$.

Define now subsets ∂_k of $\mathbb{Q}_p$ by

$$\partial_k = U_k \setminus U_{k+1}, \text{ for all } k \in \mathbb{Z}. \tag{2.4}$$

We call such μ_p-measurable subsets ∂_k, the *k-th boundaries of U_k* in $\mathbb{Q}_p$, for all $k \in \mathbb{Z}$. By (2.3) and (2.4), one obtains that

$$\mathbb{Q}_p = \bigsqcup_{k \in \mathbb{Z}} \partial_k,$$

where $\sqcup$ means the *disjoint union*, and

$$\mu_p\left(\partial_k\right) = \mu_p\left(U_k\right) - \mu_p\left(U_{k+1}\right) = \frac{1}{p^k} - \frac{1}{p^{k+1}}, \tag{2.5}$$

for all $k \in \mathbb{Z}$.

Now, let $\mathcal{M}_p$ be the set of all μ_p-measurable functions on $\mathbb{Q}_p$, i.e.,

$$\mathcal{M}_p = \mathbb{C}\left[\{\chi_S : S \in \sigma(\mathbb{Q}_p)\}\right], \tag{2.6}$$

where $\mathbb{C}[X]$ mean the algebras generated by X, understood as algebras consisting of all polynomials in X, for all sets X. So, $f \in \mathcal{M}_p$ if and only if

$$f = \sum_{S\in\sigma(\mathbb{Q}_p)} t_S\chi_S \text{ with } t_S \in \mathbb{C},$$

where $\sum$ means the *finite sum*, and χ_S are the usual *characteristic functions* of $S \in \sigma\left(\mathbb{Q}_p\right)$.

Then it forms a $*$-*algebra over* $\mathbb{C}$. Indeed, the set $\mathcal{M}_p$ of (2.6) is an algebra under the usual functional addition, and functional multiplication. Also, this algebra $\mathcal{M}_p$ has the *adjoint*,

$$\left(\sum_{S\in\sigma(G_p)} t_S\chi_S\right)^* \overset{def}{=} \sum_{S\in\sigma(G_p)} \overline{t_S}\chi_S,$$

where $t_S \in \mathbb{C}$, having their *conjugates* $\overline{t_S}$ in $\mathbb{C}$.

Let f be an element of the $*$-algebra $\mathcal{M}_p$ of (2.6). Then one can define the *p-adic integral of f* by

$$\int_{\mathbb{Q}_p} f d\mu_p = \sum_{S\in\sigma(\mathbb{Q}_p)} t_S\mu_p(S). \tag{2.7}$$

Note that, by (2.5), if $S \in \sigma(\mathbb{Q}_p)$, then there exists a subset Λ_S of $\mathbb{Z}$, such that

$$\Lambda_S = \{j \in \mathbb{Z} : S \cap \partial_j \neq \varnothing\}, \tag{2.8}$$

satisfying

$$\int_{\mathbb{Q}_p} \chi_S d\mu_p = \int_{\mathbb{Q}_p} \sum_{j\in\Lambda_S} \chi_{S\cap(x_j+\partial_j)} d\mu_p = \sum_{j\in\Lambda_S} \mu_p\left(S\cap\partial_j\right)$$

by (2.7)

$$\leq \sum_{j\in\Lambda_S} \mu_p\left(\partial_j\right) = \sum_{j\in\Lambda_S} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right),$$

by (2.5), i.e.,

$$\int_{\mathbb{Q}_p} \chi_S d\mu_p \leq \sum_{j\in\Lambda_S} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \tag{2.9}$$

for all $S \in \sigma(\mathbb{Q}_p)$, where Λ_S is in the sense of (2.8).

More precisely, one can get the following proposition.

Proposition 2.1 ([6]). *Let $S \in \sigma(\mathbb{Q}_p)$, and let $\chi_S \in \mathcal{M}_p$. Then there exist $r_j \in \mathbb{R}$, such that*

$$0 \leq r_j \leq 1 \; in \, \mathbb{R},$$

and

$$\int_{\mathbb{Q}_p} \chi_S d\mu_p = \sum_{j \in \Lambda_S} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right). \tag{2.10}$$

2.3. THE ADELE RING AND THE FINITE ADELE RING

In this section, we introduce the *Adele ring* $\mathbb{A}_{\mathbb{Q}}$, and the *finite Adele ring* $A_{\mathbb{Q}}$. For more information about $\mathbb{A}_{\mathbb{Q}}$, $A_{\mathbb{Q}}$ and the corresponding analysis, see [13].

Definition 2.2. Let $\mathcal{P}_\infty = \mathcal{P} \cup \{\infty\}$, and identify $\mathbb{Q}_\infty$ with the Banach field $\mathbb{R}$, equipped with the usual-(distance-)metric topology. Let $\mathbb{A}_{\mathbb{Q}}$ be a set

$$\mathbb{A}_{\mathbb{Q}} = \left\{ (x_p)_{p \in \mathcal{P}_\infty} \,\middle|\, \begin{array}{c} x_p \in \mathbb{Q}_p \text{ for each } p \in \mathcal{P}_\infty, \text{where only finitely many } x_q\text{'s} \\ \text{are in } \mathbb{Q}_q \setminus \mathbb{Z}_q, \text{ but all other } x_p\text{'s are contained in } \mathbb{Z}_p \text{ of } \mathbb{Q}_p \end{array} \right\}, \tag{2.11}$$

equipped with the addition $(+)$,

$$(x_p)_{p \in \mathcal{P}_\infty} + (y_p)_{p \in \mathcal{P}_\infty} = (x_p + y_p)_{p \in \mathcal{P}_\infty}, \tag{2.12}$$

and the multiplication $(\cdot)$,

$$(x_p)_{p \in \mathcal{P}_\infty} (y_p)_{p \in \mathcal{P}_\infty} = (x_p y_p)_{p \in \mathcal{P}_\infty}, \tag{2.13}$$

where $\mathbb{Z}_p$ is the unit disk of $\mathbb{Q}_p$ in the sense of Section 2, and where the entries $x_p + y_p$ of (2.12), and the entries $x_p y_p$ of (2.13) are the p-adic additioin, respectively, the p-adic multiplication on the p-adic number field $\mathbb{Q}_p$, for all $p \in \mathcal{P}$, and where $x_\infty + y_\infty$, and $x_\infty y_\infty$ are the usual $\mathbb{R}$-addition, respectively, the usual $\mathbb{R}$-multiplication.

The Adele ring $\mathbb{A}_{\mathbb{Q}}$ is equipped with the product topology of the p-adic-norm topologies for $\mathbb{Q}_p$'s, for all $p \in \mathcal{P}$, and the usual-distance-metric topology of $\mathbb{Q}_\infty = \mathbb{R}$, satisfying that

$$\left| (x_p)_{p \in \mathcal{P}_\infty} \right|_{\mathbb{Q}} = \prod_{p \in \mathcal{P}_\infty} |x_p|_p, \tag{2.14}$$

where $|\cdot|_p$ are the p-adic norms on $\mathbb{Q}_p$, for all $p \in \mathcal{P}$, and $|\cdot|_\infty$ is the usual absolute value $|\cdot|$ on $\mathbb{R} = \mathbb{Q}_\infty$.

From the above definition, the set $\mathbb{A}_{\mathbb{Q}}$ of (2.11) forms a *ring* algebraically, equipped with the binary operations (2.12) and (2.13); and this ring $\mathbb{A}_{\mathbb{Q}}$ is a *Banach space* under its $|\cdot|_{\mathbb{Q}}$-norm of (2.14), by (2.11). Thus, the set $\mathbb{A}_{\mathbb{Q}}$ of (2.11) forms a *Banach ring* induced by the family

$$\mathcal{Q} = \{\mathbb{Q}_p\}_{p \in \mathcal{P}} \cup \{\mathbb{Q}_\infty = \mathbb{R}\}.$$

Indeed, let $X = (x_p)_{p \in \mathcal{P}_\infty} \in \mathbb{A}_\mathbb{Q}$, and assume that there are $p_1, \ldots, p_N \in \mathcal{P}_\infty$, for some $N \in \mathbb{N}$, such that

$$x_{p_l} \in \mathbb{Q}_p \setminus \mathbb{Z}_p,$$

for all $l = 1, \ldots, N$, and

$$x_q \in \mathbb{Z}_q,$$

for all $q \in \mathcal{P}_\infty \setminus \{p_1, \ldots, p_N\}$. Then, by (2.11) and (2.14),

$$|x|_\mathbb{Q} = \left(\prod_{l=1}^{N} |x_{p_l}|_{p_l}\right)\left(\prod_{q \in \mathcal{P}_\infty \setminus \{p_1, \ldots, p_N\}} |x_q|_q\right) = \left(\prod_{l=1}^{N} |x_{p_l}|_{p_l}\right) \cdot 1 = \left(\prod_{l=1}^{N} |x_{p_l}|_{p_l}\right) < \infty.$$

From the definition (2.11), the Adele ring $\mathbb{A}_\mathbb{Q}$ is in fact the *weak-direct product of* $\mathcal{Q}$, expressed by

$$\mathbb{A}_\mathbb{Q} = \prod_{p \in \mathcal{P}_\infty}{}' \mathbb{Q}_p \tag{2.15}$$

(e.g., [10] and [13]), where $\prod'$ means the *weak-direct product* (satisfying the conditions of (2.11)). So, the Adele ring $\mathbb{A}_\mathbb{Q}$ can be re-defined by the weak-direct product (2.15) of the family $\mathcal{Q}$ equipped with the norm (2.14).

Definition 2.3. Let $\mathbb{A}_\mathbb{Q}$ be the Adele ring (2.11), or (2.15). Define a ring $A_\mathbb{Q}$ by

$$A_\mathbb{Q} = \left\{ (x_p)_{p \in \mathcal{P}} \;\middle|\; x_p \in \mathbb{Q}_p, \text{ for all } p \in \mathcal{P}, \text{ and } \left(0, (x_p)_{p \in \mathcal{P}}\right) \in \mathbb{A}_\mathbb{Q} \right\}, \tag{2.16}$$

set-theoretically, equipped with the inherited operations of $\mathbb{A}_\mathbb{Q}$, under subspace topology. Then this topological ring $A_\mathbb{Q}$ of the Adele ring $\mathbb{A}_\mathbb{Q}$ is said to be the finite Adele ring.

By (2.15) and (2.16), one can conclude that

$$A_\mathbb{Q} = \prod_{p \in \mathcal{P}}{}' \mathbb{Q}_p, \tag{2.17}$$

where $\prod'$ means the weak-direct product.

As in [13], one can understand the Adele ring $\mathbb{A}_\mathbb{Q}$ as a measure space equipped with the *product measure*,

$$\mu_\mathbb{Q} = \underset{p \in \mathcal{P}_\infty}{\times} \mu_p$$

where μ_p are the Haar measures on $\mathbb{Q}_p$, for all $p \in \mathcal{P}$, and μ_∞ is the *usual Lebesgue measure* on $\mathbb{Q}_\infty = \mathbb{R}$. So, the finite Adele ring $A_\mathbb{Q}$ of (2.16) can be regarded as a measure space equipped with the measure

$$\mu = \underset{p \in \mathcal{P}}{\times} \mu_p \tag{2.18}$$

on the σ-algebra $\sigma(A_\mathbb{Q})$ of $A_\mathbb{Q}$.

Similar to the $*$-algebras $\mathcal{M}_p$, for $p \in \mathcal{P}$, one can define the $*$-algebra $\mathcal{M}_{\mathcal{P}}$ by

$$\mathcal{M}_{\mathcal{P}} = \mathbb{C}\left[\{\chi_Y : Y \in \sigma(A_{\mathbb{Q}})\}\right], \tag{2.19}$$

where μ is the measure (2.18) on the finite Adele ring $A_{\mathbb{Q}}$.

By the definition (2.19), $f \in \mathcal{M}_{\mathcal{P}}$ if and only if

$$f = \sum_{Y \in \sigma(A_{\mathbb{Q}})} s_Y \chi_Y, \text{ with } s_Y \in \mathbb{C}, \tag{2.20}$$

where $\sum$ means the finite sum. Thus, one obtains the (*finite-*)*Adelic integration of* $f \in \mathcal{M}_{\mathcal{P}}$ by

$$\int_{A_{\mathbb{Q}}} f d\mu = \sum_{Y \in \sigma(A_{\mathbb{Q}})} t_Y \mu(Y), \tag{2.21}$$

whenever f is in the sense of (2.20) in $\mathcal{M}_{\mathcal{P}}$.

By the Adelic integration (2.21), one can naturally define a linear functional φ on $\mathcal{M}_{\mathcal{P}}$ by

$$\varphi(f) = \int_{A_{\mathbb{Q}}} f d\mu. \tag{2.22}$$

Equivalently, one can have a free probability space $(\mathcal{M}_{\mathcal{P}}, \varphi)$ in the sense of [12] and [14].

Definition 2.4. Let $\mathcal{M}_{\mathcal{P}}$ be the $*$-algebra (2.19), and let φ be the linear functional (2.22) on $\mathcal{M}_{\mathcal{P}}$. Then the free probability space $(\mathcal{M}_{\mathcal{P}}, \varphi)$ is called the finite-Adelic ($*$-)probability space.

Remark 2.5. Remark that the term, finite-Adelic "probability" space $(\mathcal{M}_{\mathcal{P}}, \varphi)$, does not mean it is a "probability-measure-theoretic" object. Moreover, since $\mathcal{M}_{\mathcal{P}}$ is commutative, the pair $(\mathcal{M}_{\mathcal{P}}, \varphi)$ is not a traditional (noncommutative-)free-probability-theoretic structure, either. However, the construction of the mathematical pair $(\mathcal{M}_{\mathcal{P}}, \varphi)$ is followed by the definition of (noncommutative) free probability spaces in free probability theory (well-covering commutative cases). To emphasize the construction, and to use this structure in our future research, we regard $(\mathcal{M}_{\mathcal{P}}, \varphi)$ as a free probability space, and name it the finite-Adelic probability space, even though it is not traditional both in the measure-theoretic analysis and in free probability theory.

Recall that our finite Adele ring $A_{\mathbb{Q}}$ is a weak-direct product of $\{\mathbb{Q}_p\}_{p \in \mathcal{P}}$ by (2.17), i.e.,

$$A_{\mathbb{Q}} = \prod_{p \in \mathcal{P}}{}' \mathbb{Q}_p,$$

and hence, $Y \in \sigma(A_{\mathbb{Q}})$, if and only if there exist $N \in \mathbb{N}$, and $p_1, \ldots, p_N \in \mathcal{P}$, such that

$$Y = \prod_{p \in \mathcal{P}} S_p, \text{ where } S_p \in \sigma(\mathbb{Q}_p), \tag{2.23}$$

with

$$S_p = \begin{cases} S_p \subset \mathbb{Q}_p & \text{if } p \in \{p_1, \ \ldots, \ p_N\}, \\ \mathbb{Z}_p & \text{otherwise}, \end{cases}$$

for all $p \in \mathcal{P}$. Thus, if $Y \in \sigma(A_{\mathbb{Q}})$, then

$$\varphi(\chi_Y) = \int_{A_{\mathbb{Q}}} \chi_Y \, d\mu = \int_{A_{\mathbb{Q}}} \chi_{\prod_{p \in \mathcal{P}} S_p} d\mu$$

$$= \mu\left(\prod_{p \in \mathcal{P}} S_p\right) = \prod_{p \in \mathcal{P}} \mu_p(S_p) = \prod_{p \in \mathcal{P}} \left(\int_{\mathbb{Q}_p} \chi_{S_p} d\mu_p\right),$$

since $\mu = \underset{p \in \mathcal{P}}{\times} \mu_p$

$$= \left(\prod_{l=1}^{N} \mu_{p_l}(S_p)\right)\left(\prod_{q \in \mathcal{P} \setminus \{p_1, \ldots, p_N\}} \mu_q(\mathbb{Z}_q)\right)$$

by (2.23)

$$= \prod_{l=1}^{N} \mu_{p_l}(S_p) = \prod_{l=1}^{N} \left(\varphi_{p_l}\left(\chi_{S_{p_l}}\right)\right), \tag{2.24}$$

since $\mu_q(\mathbb{Z}_q) = 1$, for all $q \in \mathcal{P}$.

Proposition 2.6. *Let $Y \in \sigma(A_{\mathbb{Q}})$ satisfy (2.23), and let χ_Y be a free random variable in our finite-Adelic probability space $(\mathcal{M}_{\mathcal{P}}, \varphi)$. Then*

$$\varphi(\chi_Y^n) = \prod_{l=1}^{N} \left(\sum_{j \in \Lambda_{S_{p_l}}} r_j^{S_{p_l}} \left(\frac{1}{p_l^j} - \frac{1}{p_l^{j+1}}\right)\right), \tag{2.25}$$

for all $n \in \mathbb{N}$, where $r_j^{S_{p_l}}$ are in the sense of (2.10), for all $j \in \Lambda_{S_{p_l}}$, for all $l = 1, \ldots, N$.

Proof. Let Y be a μ-measurable subset of $A_{\mathbb{Q}}$ satisfying (2.23). Then the μ-measurable function χ_Y satisfies that

$$\chi_Y^n = \chi_{\underbrace{Y \cap Y \cap \ldots\ldots \cap Y}_{n\text{-times}}} = \chi_Y,$$

for all $n \in \mathbb{N}$. So,

$$\varphi(\chi_Y^n) = \varphi(\chi_Y) = \prod_{l=1}^{N} \left(\varphi_{p_l}\left(\chi_{S_{p_l}}\right)\right),$$

by (2.24), for all $n \in \mathbb{N}$. Therefore, by (2.10), we obtain the free-moment formula (2.25). $\square$

By (2.25), if

$$f = \sum_{Y \in \sigma(A_{\mathbb{Q}})} s_Y \chi_Y, \text{ with } s_Y \in \mathbb{C},$$

in $\mathcal{M}_{\mathcal{P}}$, where Y satisfy (2.23), then

$$\varphi(f) = \int_{A_{\mathbb{Q}}} f d\mu = \sum_{Y \in \sigma(A_{\mathbb{Q}})} s_Y \left(\prod_{l=1}^{N_S} \left(\sum_{j \in \Lambda_{S_{p_l}^Y}} r_j^{S_{p_l}^Y} \left(\frac{1}{p_l^j} - \frac{1}{p_l^{j+1}} \right) \right) \right). \tag{2.26}$$

The formula (2.26) illustrates that the formula (2.25) provides a general tool to study finite-Adelic calculus.

Notice that

$$\mathcal{M}_{\mathcal{P}} = \prod_{p \in \mathcal{P}}' \mathcal{M}_p$$

where $\prod'$ means the *weak-direct* (or *weak-tensor*) *product of* $*$*-algebras*. The isomorphism theorem (2.27) holds because of (2.17) and (2.23).

Theorem 2.7. *Let* $(\mathcal{M}_{\mathcal{P}}, \varphi)$ *be the finite-Adelic probability space. Then*

$$\mathcal{M}_{\mathcal{P}} = \prod_{p \in \mathcal{P}}' \mathcal{M}_p, \text{ and } \varphi = \prod_{p \in \mathcal{P}} \varphi_p, \tag{2.27}$$

where $\mathcal{M}_p$ *are in the sense of* (2.6)*, and* φ_p *are the* p*-adic integrations on* $\mathcal{M}_p$*, for all* $p \in \mathcal{P}$*.*

Proof. The $*$-isomorphism theorem of $\mathcal{M}_{\mathcal{P}}$ in (2.27) is proven by (2.17) and (2.23). The equivalence for φ in (2.27) is guaranteed by (2.18) and (2.25). □

3. ANALYSIS OF $(\mathcal{M}_{\mathcal{P}}, \varphi)$

In this section, we consider functional-analytic properties on our finite-Adelic probability space $(\mathcal{M}_{\mathcal{P}}, \varphi)$. In particular, such properties are represented by the distributional data from elements of $(\mathcal{M}_{\mathcal{P}}, \varphi)$ under free-probability-theoretic language, free moments. As application, we show relations between our free moments and the Euler-totient-functional values.

Let $(\mathcal{M}_{\mathcal{P}}, \varphi)$ be the finite-Adelic probability space. From constructions, one can get that

$$\mathcal{M}_{\mathcal{P}} = \prod_{p \in \mathcal{P}}' \mathcal{M}_p, \text{ and } \varphi = \prod_{p \in \mathcal{P}} \varphi_p, \tag{3.1}$$

by (2.27), where φ_p are the p-adic integrations on $\mathbb{Q}_p$, for all $p \in \mathcal{P}$ (and hence, φ is the Adelic integration on $A_{\mathbb{Q}}$). So, by abusing notation, one may/can re-write the relations of (3.1) as follows:

$$(\mathcal{M}_{\mathcal{P}}, \varphi) = \prod_{p \in \mathcal{P}}' (\mathcal{M}_p, \varphi_p). \tag{3.2}$$

Recall that, in [5, 6] and [8], we call $(\mathcal{M}_p, \varphi_p)$ the *p-adic probability spaces*, for all $p \in \mathcal{P}$.

Here, we concentrate on computing free distributions of elements generated by generating elements of $\mathcal{M}_{\mathcal{P}}$.

Theorem 3.1. *Let $Y_1, \ldots, Y_n \in \sigma(A_{\mathbb{Q}})$, and $\chi_{Y_l} \in (\mathcal{M}_{\mathcal{P}}, \varphi)$, for $l = 1, \ldots, n$, for some $n \in \mathbb{N}$. Then there exist a unique "finite" subset P_o of $\mathcal{P}$, and*

$$X_p \in \sigma(\mathbb{Q}_p), \text{ for all } p \in P_o,$$

such that

$$\varphi\left(\prod_{l=1}^{n} \chi_{Y_l}\right) = \prod_{p \in P_o}\left(\sum_{j \in \Lambda_{X_p}} r_j^{X_p}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right)\right), \tag{3.3}$$

where $r_j^{X_p}$ are in the sense of (2.10), *and Λ_{X_p} are in the sense of* (2.8).

Proof. Let $Y_1, \ldots, Y_n$ be μ-measurable subsets of the finite Adele ring $A_{\mathbb{Q}}$, for $n \in \mathbb{N}$. So, by (2.23), for each Y_i, there exist a unique $N_i \in \mathbb{N}$, and $p_{i,1}, \ldots, p_{i,N_i} \in \mathcal{P}$, such that

$$Y_i = \prod_{p \in \mathcal{P}} S_p^i, \text{ with } S_p^i \in \sigma(\mathbb{Q}_p), \tag{3.4}$$

and

$$S_p^i = \begin{cases} S_p^i & \text{if } p \in \{p_{i,1}, \ldots, p_{i,N_i}\}, \\ \mathbb{Z}_p & \text{otherwise,} \end{cases}$$

for all $p \in \mathcal{P}$, for all $i = 1, \ldots, n$.

If we let $h = \prod_{l=1}^{N} \chi_{Y_l}$, and $h_l = \chi_{Y_l}$, for $l = 1, \ldots, n$, then

$$\varphi(h) = \varphi\left(\prod_{i=1}^{n} h_i\right) = \varphi\left(\chi_{Y_o}\right), \tag{3.5}$$

where

$$Y_o = \bigcap_{i=1}^{n} Y_i = \bigcap_{i=1}^{n}\left(\prod_{p \in \mathcal{P}} S_p^i\right) = \prod_{p \in \mathcal{P}}\left(\bigcap_{i=1}^{n} S_p^i\right) \tag{3.6}$$

in $\sigma(A_{\mathbb{Q}})$.

For the μ-measurable set Y_o of (3.6), there exists a subset P_{Y_o} of $\mathcal{P}$,

$$P_{Y_o} = \bigcup_{i=1}^{n} \{p_{i,1}, \ldots, p_{i,N_i}\} \text{ in } \mathcal{P}, \tag{3.7}$$

such that

$$\bigcap_{i=1}^{n} S_p^i = \begin{cases} \bigcap\limits_{i=1}^{n} S_p^i & \text{if } p \in P_{Y_o}, \\ \bigcap\limits_{i=1}^{n} \mathbb{Z}_p = \mathbb{Z}_p & \text{otherwise,} \end{cases} \tag{3.8}$$

for all $p \in \mathcal{P}$.

Remark that either

$$\bigcap_{i=1}^{n} S_p^i \neq \mathbb{Z}_p \quad \text{or} \quad \bigcap_{i=1}^{n} S_p^i = \mathbb{Z}_p,$$

in (3.8), case-by-case, for $p \in P_{Y_0}$. However, in general, the equality (3.8) holds with respect to P_{Y_o} of (3.7).

Therefore, the formula (3.5) goes to

$$\varphi(h) = \varphi\left(\chi_{Y_o}\right) = \prod_{p \in P_{Y_o}} \varphi_p\left(\chi_{\bigcap_{i=1}^{n} S_p^i}\right) \prod_{p \in P_{Y_o}} \mu_p\left(\bigcap_{i=1}^{n} S_p^i\right)$$

by (2.25)

$$= \prod_{p \in P_{Y_o}} \left(\sum_{j \in \Lambda_{\bigcap_{i=1}^{n} S_p^i}} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right) \right),$$

by (2.10), where r_j are in the sense of (2.10), for all $j \in \Lambda_{\bigcap_{i=1}^{n} S_p^i}$, where $\Lambda_{\bigcap_{i=1}^{n} S_p^i}$ are in the sense of (2.8), for all $p \in P_{Y_o}$ in $\mathcal{P}$.

Therefore, if we take

$$P_o = P_{Y_o} \text{ of } (3.7)$$

and

$$X_p = \bigcap_{i=1}^{n} S_p^i \text{ in } \mathbb{Q}_p \text{ for all } p \in P_{Y_o},$$

then the formula (3.3) is well-determined. □

The above joint free-moment formula (3.3) characterizes the free distributions of generating elements of our finite-Adelic probability space $(\mathcal{M}_{\mathcal{P}}, \varphi)$.

As a corollary of (2.25) and (3.3), one obtains the following result.

Corollary 3.2. *Let $Y \in \sigma(A_{\mathbb{Q}})$, satisfying that $Y = \prod_{p \in \mathcal{P}} S_p$ with $S_p \in \sigma\left(\mathbb{Q}_p\right)$,*

$$S_p = \begin{cases} \partial_{k_l}^{p_l} & \text{if } p = p_l, \text{ for } l = 1, \ldots, N, \\ \mathbb{Z}_p & \text{otherwise,} \end{cases} \tag{3.9}$$

for all $p \in \mathcal{P}$, for some $p_1, \ldots, p_N \in \mathcal{P}$, for $k_1, \ldots, k_N \in \mathbb{Z}$, for $N \in \mathbb{N}$, where $\partial_{k_l}^{p_l}$ are the k_l-th boundary of basis elements $U_{k_l}^{p_l} = p_l^{k_l} \mathbb{Z}_{p_l}$ of $\mathbb{Q}_{p_l}$, for all $l = 1, \ldots, N$. Then

$$\varphi\left(\chi_Y^n\right) = \prod_{l=1}^{N} \left(\frac{1}{p_l^{k_l}} - \frac{1}{p_l^{k_l+1}}\right) \tag{3.10}$$

for all $n \in \mathbb{N}$.

Proof. The proof of (3.10) is done by (2.25) and (3.3). Indeed, if Y satisfies the condition (3.9), then

$$\varphi\left(\chi_Y\right)=\prod_{l=1}^{N}\varphi_{p_l}\left(\chi_{\partial_{k_l}^{p_l}}\right)=\prod_{l=1}^{N}\left(\frac{1}{p_l^{k_l}}-\frac{1}{p_l^{k_l+1}}\right).$$

Therefore, since $\chi_Y^n=\chi_Y$, for all $n\in\mathbb{N}$, the free-moment formula (3.10) holds. $\square$

By (3.3) and (3.10), we obtain the following corollary, too.

Corollary 3.3. *Let $Y_l=\prod_{p\in\mathcal{P}}S_p^l\in\sigma(A_{\mathbb{Q}})$, for $l=1,\dots,n$, for some $n\in\mathbb{N}$, where*

$$S_p^l=\begin{cases}\partial_{k_{p_t,l}}^{p_t} & \text{if } p_t\in\{p_{l,1},\ \dots,\ p_{l,N_l}\},\\ \mathbb{Z}_p & \text{otherwise,}\end{cases} \tag{3.11}$$

where $\partial_{k_p}^p$ are the k_p-th boundaries for $k_p\in\mathbb{Z}$ in $\mathbb{Q}_p$, for $p\in\mathcal{P}$, and where $k_{p_t,1},\dots,k_{p_t,N_l}\in\mathbb{Z}$, for all $l=1,\ \dots,\ n$, all $p\in\mathcal{P}$. Now, let

$$P_o=\bigcup_{l=1}^{n}\{p_{l,1},\dots,p_{l,N_l}\}\ \text{in}\ \mathcal{P}.$$

Then one obtains that

$$\varphi\left(\prod_{l=1}^{n}\chi_{Y_l}\right)=\prod_{p\in P_o}\omega_p\left(\frac{1}{p^{k_p}}-\frac{1}{p^{k_p+1}}\right), \tag{3.12}$$

where p^{k_p} are in the sense of (3.11)*, where*

$$\omega_p=\begin{cases}1 & \text{if } \bigcap_{l=1}^{n}S_p^l\neq\varnothing,\\ 0 & \text{otherwise}\end{cases}$$

for all $p\in P_o$.

Proof. The proof of (3.12) is done by (3.3) and (3.10), under the condition (3.11). $\square$

Let $Y_l\in\sigma(A_{\mathbb{Q}})$ be in the sense of (3.11), for $l=1,\dots,n$, and let

$$X=\prod_{l=1}^{n}\chi_{Y_l}\in\left(\mathcal{M}_{\mathcal{P}},\varphi\right). \tag{3.13}$$

Such free random variables X of (3.13) are called *boundary-product elements* of the finite-Adelic probability space $(\mathcal{M}_{\mathcal{P}},\varphi)$.

As we have seen in (3.12), if X is a boundary-product element of $\left(\mathcal{M}_{\mathcal{P}},\varphi\right)$, then there exists a subset P_o of $\mathcal{P}$ such that

$$\varphi\left(X\right)=\prod_{p\in P_o}\omega_p\left(\frac{1}{p^{k_p}}-\frac{1}{p^{k_p+1}}\right),$$

for some $k_p\in\mathbb{Z}$ (in the sense of (3.10)), for all $p\in P_o$, where ω_p is in the sense of (3.12).

Definition 3.4. Let X be a boundary-product element (3.13) of $(\mathcal{M}_{\mathcal{P}}, \varphi)$. Assume now that Y_l are in the sense of (3.11), and P_o is in the sense of (3.12). Assume further that, for all $p \in P_o$, the corresponding integers k_p are nonnegative, i.e.,

$$k_p \geq 0, \text{ for all } p \in P_o. \tag{3.14}$$

Then we say this boundary-product element X is a $(+)$-boundary element of $(\mathcal{M}_{\mathcal{P}}, \varphi)$, i.e., X is a $(+)$-boundary element if and only if, (i) X is in the sense of (3.13), and (ii) the corresponding finite subset P_o of $\mathcal{P}$ satisfies the condition (3.14).

Remark 3.5. In the rest of this section, we focus on studying $(+)$-boundary elements (3.14) of the finite-Adelic probability space $(\mathcal{M}_{\mathcal{P}}, \varphi)$. As we have seen in (3.3), the free-distributional data of χ_Y, for an arbitrary μ-measurable subsets Y of $A_{\mathbb{Q}}$, are determined by the free distributions (3.10) of boundary-product elements, or those (3.12) of of their operator products. So, it is reasonable to restrict our interests to investigate free-distributional information of boundary-product elements (3.13) for studying free distributions of arbitrary elements of the finite-Adelic $*$-algebra $\mathcal{M}_{\mathcal{P}}$, under the Adelic integration φ. However, as we checked, the free-probabilisitc information (3.12) is determined by ω_p, by the chain property in (2.3), i.e.,

$$\partial_{k_p}^p \cap \mathbb{Z}_p = \begin{cases} \partial_{k_p}^p & \text{if } k_p \geq 0 \text{ in } \mathbb{Z}, \\ \varnothing & \text{if } k_p < 0 \text{ in } \mathbb{Z} \end{cases}$$

for all $p \in \mathcal{P}$, where $\varnothing$ means the empty set.

It shows that if a boundary-product element X has a finite subset

$$P_o = \{p \in \mathcal{P} : \partial_{k_p}^p \neq \mathbb{Z}_p\} \text{ of } \mathcal{P},$$

partitioned by

$$P_o = P_o^+ \sqcup P_o^-,$$

where

$$P_o^+ = \{p \in P_o : k_p \geq 0 \text{ in } \mathbb{Z}\},$$

and

$$P_o^- = \{q \in P_o : k_q < 0 \text{ in } \mathbb{Z}\},$$

equppied with $P_o^- \neq \varnothing$, then the formula (3.12) vanishes by the role of ω_q for $q \in P_o^-$.

Therefore, to avoid such vanishing cases, one had better focus on the cases where we have $(+)$-boundary elements satisfying (3.14), rather than whole boundary-product elements in $(\mathcal{M}_{\mathcal{P}}, \ \varphi)$.

As we discussed in the above remark, one can realize that such $(+)$-boundary elements provide certain building blocks of the "non-vanishing" free distribution (3.3) (under certain additional multiples). So, it is natural to concentrate on studying free distributions of such elements in the finite-Adelic probability space $(\mathcal{M}_{\mathcal{P}}, \ \varphi)$.

Let $\phi : \mathbb{N} \to \mathbb{C}$ be the *Euler totient function* defined by an *arithmetic function*,

$$\phi(n) = |\{k \in \mathbb{N} \mid 1 \leq k \leq n, \gcd(n, \ k) = 1\}|, \tag{3.15}$$

for all $n \in \mathbb{N}$, where $|S|$ mean the *cardinalities of sets* S, and gcd means the *greatest common divisor*. It is well-known that

$$\phi(n) = n \left(\prod_{p \in \mathcal{P},\ p|n} \left(1 - \frac{1}{p}\right) \right) \text{ for all } n \in \mathbb{N}, \tag{3.16}$$

where "$p \mid n$" means "p divides n," or "n is divisible by p." For instance,

$$\phi(p) = p - 1 = p\left(1 - \frac{1}{p}\right)$$

for all $p \in \mathcal{P}$, by (3.15) and (3.16).

Remark that the Euler totient function ϕ is a *multiplicative arithmetic function* in the sense that

$$\phi(n_1 n_2) = \phi(n_1)\phi(n_2), \tag{3.17}$$

whenever

$$\gcd(n_1, n_2) = 1$$

for all $n_1, n_2 \in \mathbb{N}$.

If $p_1 \neq p_2$ in $\mathcal{P}$, then, for any $n_1, n_2 \in \mathbb{N}$,

$$\gcd\left(p_1^{n_1},\ p_2^{n_2}\right) = 1,$$

and hence,

$$\phi\left(p_1^{n_1} p_2^{n_2}\right) = \phi\left(p_1^{n_1}\right)\phi\left(p_2^{n_2}\right) = p_1^{n_1} p_2^{n_2}\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p^2}\right),$$

by (3.16) and (3.17).

Theorem 3.6. *Let X be a $(+)$-boundary element* (3.13) *of the finite-Adelic probability space $(\mathcal{M}_{\mathcal{P}}, \varphi)$ satisfying* (3.14). *Then there exist the subset P_o of $\mathcal{P}$, and*

$$K_o = \{k_p \in \mathbb{N}_0 : p \in P_o\} \text{ of } \mathbb{Z},$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, such that

$$\begin{aligned} n_X &= \prod_{p \in P_o} p^{k_p} \in \mathbb{N}, \\ 0 < r_X &= \prod_{p \in P_o} \frac{1}{p^{k_p+1}} \leq 1 \text{ in } \mathbb{Q}, \\ \varphi(X) &= r_X \phi(n_X). \end{aligned} \tag{3.18}$$

Proof. Let X be a $(+)$-boundary element (3.14) in $(\mathcal{M}_{\mathcal{P}}, \varphi)$. Then, by (3.12), there exist the subsets P_o of $\mathcal{P}$, and K_o of $\mathbb{Z}$, such that

$$\varphi(X) = \prod_{p \in P_o} \left(\frac{1}{p^{k_p}} - \frac{1}{p^{k_p+1}}\right),$$

with $k_p \in K_o$, with $k_p \geq 0$ in $\mathbb{Z}$.

Observe that

$$\begin{aligned}
\varphi(X) &= \prod_{p\in P_o} \frac{1}{p^{k_p}}\left(1-\frac{1}{p}\right) \\
&= \left(\prod_{p\in P_o} \frac{p^{k_p}}{p^{k_p+1}}\left(1-\frac{1}{p}\right)\right) \\
&= \left(\prod_{p\in P_o} \frac{1}{p^{k_p+1}}\right)\left(\prod_{p\in P_o} p^{k_p}\left(1-\frac{1}{p}\right)\right) \\
&= \left(\prod_{p\in P_o^+} \frac{1}{p^{k_p+1}}\right)\phi\left(n_X\right),
\end{aligned}$$

where

$$n_X = \left(\prod_{p\in P_o} p^{k_p}\right) \in \mathbb{N},$$

by (3.15), (3.16) and (3.17). □

The above theorem, especially, the relations in (3.18), characterizes the free distributions of (+)-boundary elements of $(\mathcal{M}_{\mathcal{P}}, \varphi)$, in terms of the Euler-totient-functional values. So, it illustrates the connections between our free-probabilistic structure and number-theoretic results.

If X is a (+)-boundary element (3.14) in $(\mathcal{M}_{\mathcal{P}},\ \varphi)$, then there exist

$$0 < r_X \leq 1 \text{ in } \mathbb{Q}, \text{ and } n_X \in \mathbb{N},$$

such that

$$\varphi(X) = r_X\phi(n_X) \quad \Leftrightarrow \quad \phi(n_X) = \frac{\varphi(X)}{r_X},$$

by (3.18). Observe now a converse of the above theorem.

Theorem 3.7. *Let $n \in \mathbb{N}$ be prime-factorized by*

$$n = p_1^{k_{p_1}} p_2^{k_{p_2}} \dots p_N^{k_{p_N}} \text{ in } \mathbb{N}, \tag{3.19}$$

where $p_1, \dots, p_N \in \mathcal{P}$, and $k_{p_1}, \dots, k_{p_N} \in \mathbb{N}$, for some $N \in \mathbb{N}$. Then there exists a (+)-boundary element X of the finite-Adelic probability space $(\mathcal{M}_{\mathcal{P}}, \varphi)$ such that

$$X = \prod_{p\in\mathcal{P}} \chi_{Y_p} \in \left(\mathcal{M}_{\mathcal{P}}, \varphi\right), \tag{3.20}$$

having

$$P_o = \{p_1, \dots, p_N\} \subset \mathcal{P},$$

$$K_o = \{k_{p_1}, \dots, k_{p_N}\} \subset \mathbb{N}_0,$$

and

$$Y_p = \begin{cases} \partial^p_{k_p} & \textit{if } p \in P_o, \\ \mathbb{Z}_p & \textit{otherwise} \end{cases}$$

for all $p \in \mathcal{P}$, *satisfying that*

$$\phi\,(n) = n_p n \varphi(X), \ \textit{with } n_p = \prod_{p \in P_o} p \in \mathbb{N}. \tag{3.21}$$

Proof. Let X be a $(+)$-boundary element (3.20) in $(\mathcal{M}_{\mathcal{P}}, \varphi)$. Then

$$\varphi\,(X) = \left(\prod_{p \in P_o} \varphi_p \left(\chi_{\partial^p_{k_p}}\right)\right) = \left(\prod_{p \in P_o} \frac{1}{p^{k_p}} \left(1 - \frac{1}{p}\right)\right)$$
$$= \left(\prod_{p \in P_o} \frac{1}{p^{k_p+1}}\right) \left(\prod_{p \in P_o^+} p^{k_p} \left(1 - \frac{1}{p}\right)\right)$$

like in the proof of (3.18)

$$= \left(\prod_{p \in P_o} \frac{1}{p^{k_p}}\right) \left(\prod_{p \in P_o} \frac{1}{p}\right) (\phi(n))$$

where n is given as above in $\mathbb{N}$

$$= \left(\frac{1}{\prod_{p \in P_o} p^{k_p}}\right) \left(\prod_{p \in P_o} \frac{1}{p}\right) (\phi(n))$$
$$= \left(\frac{1}{n}\right) \left(\frac{1}{\prod_{p \in P_o^+} p}\right) (\phi(n))\,.$$

Therefore, for any $n \in \mathbb{N}$, with its prime-factorization (3.19), there exists a $(+)$-boundary element X of (3.20), such that

$$\varphi(X) = \frac{1}{n_p n} \phi(n), \ \text{with } n_p = \prod_{p \in P_o} p \in \mathbb{N}.$$

Therefore, one can obtain the relation (3.21), whenever n of (3.19) is fixed in $\mathbb{N}$. □

The above two theorems provide a connection between number-theoretic results from the Adelic analysis and arithmetic function theory, and our free-probabilistic results on the $*$-algebra $\mathcal{M}_{\mathcal{P}}$. In particular, the above two theorems show that: $n \in \mathbb{N}$ if and only if there exists $X \in (\mathcal{M}_{\mathcal{P}}, \varphi)$ such that

$$\varphi(X) = n_o \frac{\phi(n)}{n} \text{ for some } n_o \in \mathbb{N},$$

by (3.18) and (3.21).

4. REPRESENTATION OF $(\mathcal{M}_{\mathcal{P}},\ \varphi)$

Let $(\mathcal{M}_{\mathcal{P}}, \varphi)$ be the finite-Adelic probability space,

$$(\mathcal{M}_{\mathcal{P}}, \varphi) = \prod_{p \in \mathcal{P}}{}' (\mathcal{M}_p, \varphi_p) = \left(\prod_{p \in \mathcal{P}}{}' \mathcal{M}_p,\ \prod_{p \in \mathcal{P}} \varphi_p \right). \tag{4.1}$$

In [5,6] and [8], we established and studied Hilbert-space representations $(\mathfrak{H}_p,\ \alpha^p)$ of the $*$-probability spaces $(\mathcal{M}_p, \varphi_p)$, for $p \in \mathcal{P}$. By (4.1), one can construct a Hilbert-space representation of $\mathcal{M}_{\mathcal{P}}$ with help of the representations,

$$(\mathfrak{H}_p,\ \alpha^p) \text{ of } \mathcal{M}_p' s, \text{ for all } p \in \mathcal{P}.$$

Define a form

$$[\cdot, \cdot] : \mathcal{M}_{\mathcal{P}} \times \mathcal{M}_{\mathcal{P}} \to \mathbb{C}$$

by

$$[f_1, f_2] \overset{def}{=} \int\limits_{A_{\mathbb{Q}}} f_1 f_2^* d\mu, \quad f_1, f_2 \in \mathcal{M}_{\mathcal{P}}. \tag{4.2}$$

Then, by the definition (4.2), this form $[\cdot, \cdot]$ is *sesqui-linear*:

$$[t_1 f_1 + t_2 f_2, f_3] = t_1 [f_1, f_3] + t_2 [f_2, f_2]$$

and

$$[f_1, t_1 f_2 + t_2 f_3] = \overline{t_1} [f_1, f_2] + \overline{t_2} [f_1, f_2] \tag{4.3}$$

for all $t_1, t_2 \in \mathbb{C}$ and $f_1, f_2, f_3 \in \mathcal{M}_{\mathcal{P}}$. Now, observe that

$$[f_1, f_2] = \int\limits_{A_{\mathbb{Q}}} f_1 f_2^* d\mu = \int\limits_{A_{\mathbb{Q}}} (f_2 f_1^*)^* \, d\mu = \overline{\int\limits_{A_{\mathbb{Q}}} f_2 f_1^* \, d\mu} = \overline{[f_2,\ f_1]} \tag{4.4}$$

for all $f_1, f_2 \in \mathcal{M}_{\mathcal{P}}$.

Let $Y \in \sigma(A_{\mathbb{Q}})$ and $t \in \mathbb{C}$, inducing $f = t\chi_Y \in \mathcal{M}_{\mathcal{P}}$. Then

$$[f, f] = \int\limits_{A_{\mathbb{Q}}} f f^* d\mu = \int\limits_{A_{\mathbb{Q}}} |t|^2 \chi_Y d\mu = |t|^2 \mu(Y) \geq 0,$$

where $|t|$ means the *omdulus of* t in $\mathbb{C}$, and hence,

$$[h,h] \geq 0, \text{ for all } h \in \mathcal{M}_{\mathcal{P}} \tag{4.5}$$

For $f = t\chi_Y \in \mathcal{M}_{\mathcal{P}}$, assume that

$$[f,f] = 0 \Longleftrightarrow |t|^2 \mu(Y) = 0$$
$$\Longleftrightarrow |t|^2 = 0 \text{ or } \mu(Y) = 0 \Longleftrightarrow t = 0 \text{ or } Y = \varnothing, \text{ the empty set in } A_{\mathbb{Q}},$$

because $\mu = \underset{p\in\mathcal{P}}{\times} \mu_p$, and μ_p are the Haar measures on $\mathbb{Q}_p$, for all $p \in \mathcal{P}$ (e.g., see (4.1))

$$\Longleftrightarrow f = 0 \cdot \chi_Y = O \text{ or } f = t\chi_{\varnothing} = O,$$

where O means the *zero element of* $\mathcal{M}_{\mathcal{P}}$.

In other words,

$$[t\chi_Y, t\chi_Y] = 0 \text{ if and only if } t\chi_Y = O \text{ in } \mathcal{M}_{\mathcal{P}}.$$

Therefore, one has

$$[f,f] = 0 \Longleftrightarrow f = O \text{ in } \mathcal{M}_{\mathcal{P}}. \tag{4.6}$$

Proposition 4.1. *The form* $[\cdot,\cdot]$ *of* (4.2) *on the finite-Adelic $*$-algebra* $\mathcal{M}_{\mathcal{P}}$ *is an inner product. Equivalently, the pair* $(\mathcal{M}_{\mathcal{P}},\ [\cdot,\cdot])$ *forms an inner product space.*

Proof. The form $[\cdot,\cdot]$ of (4.2) is an inner product on $\mathcal{M}_{\mathcal{P}}$, because it satisfies (4.3), (4.4), (4.5) and (4.6). □

Let $[\cdot,\cdot]$ be the inner product (4.2) on the $*$-algebra $\mathcal{M}_{\mathcal{P}}$. Define now the norm $\|\cdot\|$ and the metric d on $\mathcal{M}_{\mathcal{P}}$ by

$$\|f\| = \sqrt{[f,\ f]}, \text{ for all } f \in \mathcal{M}_{\mathcal{P}}, \tag{4.7}$$

respectively

$$d(f_1, f_2) = \|f_1 - f_2\|, \text{ for all } f_1, f_2 \in \mathcal{M}_{\mathcal{P}}. \tag{4.8}$$

Definition 4.2. Let d be the metric (4.8) induced by the norm $\|\cdot\|$ of (4.7) on the inner product space $(\mathcal{M}_{\mathcal{P}},\ [\cdot,\cdot])$. Then maximal d-metric-topology closure in $\mathcal{M}_{\mathcal{P}}$ is called the finite-Adelic Hilbert space, and we denote it by $H_{\mathcal{P}}$.

By the very definitioin of finite-Adelic Hilbert space $H_{\mathcal{P}}$, the $*$-algebra $\mathcal{M}_{\mathcal{P}}$ is acting on $H_{\mathcal{P}}$ via a linear morphism $\alpha : \mathcal{M}_{\mathcal{P}} \to B(H_{\mathcal{P}})$,

$$\alpha(f)(h) = fh, \text{ for all } h \in H_{\mathcal{P}}, \tag{4.9}$$

for all $f \in \mathcal{M}_{\mathcal{P}}$, i.e., the algebra-action α of (4.9) assigns each element f of $\mathcal{M}_{\mathcal{P}}$ to the *multiplication operator* $\alpha(f)$ *with its symbol* f in the *operator algebra* $B(H_{\mathcal{P}})$ (consisting of all bounded linear operators on $H_{\mathcal{P}}$).

Notation 4.3. For convenience, we denote $\alpha(f)$ by α_f, for all $f \in \mathcal{M}_{\mathcal{P}}$, where α is in the sense of (4.9). Moreover, let us denote $\alpha(\chi_Y) = \alpha_{\chi_Y}$ simply by α_Y, for all $Y \in \sigma(A_{\mathbb{Q}})$.

By the definition (4.9), for any $f_1, f_2 \in \mathcal{M}_{\mathcal{P}}$, one has

$$\alpha_{f_1 f_2} = \alpha_{f_1}\alpha_{f_2} \text{ on } H_{\mathcal{P}}, \tag{4.10}$$

and

$$(\alpha_f)^* = \alpha_{f^*} \text{ for all } f \in \mathcal{M}_{\mathcal{P}}. \tag{4.11}$$

Theorem 4.4. *Let $H_{\mathcal{P}}$ be the finite-Adelic Hilbert space, and let α be in the sense of* (4.9)*. Then the pair $(H_{\mathcal{P}},\ \alpha)$ is a Hilbert-space representation of the finite-Adelic $*$-algebra $\mathcal{M}_{\mathcal{P}}$.*

Proof. It suffices to show that the linear morphism α of (4.9) is a well-determined $*$-homomorphism from $\mathcal{M}_{\mathcal{P}}$ to the operator algebra $B(H_{\mathcal{P}})$. Note that, by (4.10) and (4.11), α is indeed a $*$-homomorphism from $\mathcal{M}_{\mathcal{P}}$ to $B\left(H_{\mathcal{P}}\right)$. □

By the above theorem, one can understand all elements f of $\mathcal{M}_{\mathcal{P}}$ as a *Hilbert-space operator α_f on $H_{\mathcal{P}}$*.

Definition 4.5. Let $(\mathcal{M}_{\mathcal{P}}, \varphi)$ be the finite-Adelic probability space, and let $(H_{\mathcal{P}},\ \alpha)$ be the representation of $\mathcal{M}_{\mathcal{P}}$ of the above theorem. Then we call this representation, the finite-Adelic representation of $\mathcal{M}_{\mathcal{P}}$. Define now the C^*-subalgebra $M_{\mathcal{P}}$ of the operator algebra $B\left(H_{\mathcal{P}}\right)$ by

$$M_{\mathcal{P}} = C^*\left(\mathcal{M}_{\mathcal{P}}\right) \overset{def}{=} \overline{\mathbb{C}\left[\alpha\left(\mathcal{M}_{\mathcal{P}}\right)\right]}, \tag{4.12}$$

where $\overline{X}$ mean the operator-norm-topology closures of subsets X of $B\left(H_{\mathcal{P}}\right)$. We call this C^*-subalgebra, the finite-Adelic C^*-algebra.

5. FUNCTIONAL-ANALYTIC PROPERTIES ON $M_{\mathcal{P}}$

In this section, we study functional-analytic properties on the finite-Adelic C^*-algebra $M_{\mathcal{P}}$ of (4.12) under suitable free-probabilistic models. Such properties are determined by the analytic data of Section 3, implying number-theoretic information.

Let $(\mathcal{M}_{\mathcal{P}}, \varphi)$ be the finite-Adelic probability space, and let $(H_{\mathcal{P}},\ \alpha)$ be the finite-Adelic representation of $\mathcal{M}_{\mathcal{P}}$. Let $M_{\mathcal{P}}$ be the finite-Adelic C^*-algebra (4.12) of $(\mathcal{M}_{\mathcal{P}}, \varphi)$ under $(H_{\mathcal{P}},\ \alpha)$. In this section, we will consider free-probabilistic data on the C^*-algebra $M_{\mathcal{P}}$ by constructing a system of suitable linear functionals on $M_{\mathcal{P}}$.

Define a linear functional $\varphi_{p,j}$ on $M_{\mathcal{P}}$ by

$$\varphi_{p,j}(T) = \left[T\left(\chi_{B_j^p}\right),\ \chi_{B_j^p}\right], \text{ for all } T \in M_{\mathcal{P}}, \tag{5.1}$$

for all $p \in \mathcal{P}, j \in \mathbb{N}_0$, where

$$\mathbb{N}_0 \overset{def}{=} \mathbb{N} \cup \{0\},$$

and

$$B_j^p = \prod_{q \in \mathcal{P}} Y_q \text{ in } \sigma\left(A_{\mathbb{Q}}\right)$$

with

$$Y_q = \begin{cases} \partial_j^p & \text{if } q = p, \\ \mathbb{Z}_q & \text{otherwise} \end{cases}$$

for all $q \in \mathcal{P}$, i.e.,

$$\chi_{B_j^p} = \chi_{\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \ldots \times \underset{p\text{-th position}}{\partial_j^p} \times \ldots} \in H_{\mathcal{P}}$$

for $p \in \mathcal{P}$ and $j \in \mathbb{N}_0$.

Remark 5.1. In the definition (5.1), remark that we only take j from $\mathbb{N}_0$, not from $\mathbb{Z}$. The reason is as follows: Let $Y \in \sigma(A_{\mathbb{Q}})$, with

$$Y = \prod_{q \in \mathcal{P}} S_q, \text{ where } S_q \in \sigma\left(\mathbb{Q}_q\right),$$

where there exists a finite subset

$$P_Y = \{p_1, \ldots, p_N\} \text{ of } \mathcal{P}, \text{ for some } N \in \mathbb{N},$$

satisfying

$$P_Y = P_Y^+ \sqcup P_Y^-$$

such that

$$P_Y^+ = \{q \in P_Y : k_q \geq 0 \text{ in } \mathbb{Z}\},$$

and

$$P_Y^- = \{q \in P_Y : k_q < 0 \text{ in } \mathbb{Z}\},$$

where

$$S_q = \begin{cases} \partial_{k_q}^q & \text{if } q \in P_Y, \\ \mathbb{Z}_q & \text{otherwise} \end{cases}$$

for all $q \in \mathcal{P}$. (See (5.6) below.)

Also, let B_j^p be in the sense of (5.1), where $p \in \mathcal{P}$, and "$j \in \mathbb{Z}$." Assume first that $j < 0$ in $\mathbb{Z}$, and $p \notin P_Y$ in $\mathcal{P}$. Then

$$\begin{aligned} Y \cap B_j^p &= (S_2 \cap \mathbb{Z}_2) \times (S_3 \cap \mathbb{Z}_3) \times \ldots \times \underset{p\text{-th position}}{\left(\mathbb{Z}_p \cap \partial_j^p\right)} \times \ldots \\ &= (S_2 \cap \mathbb{Z}_2) \times (S_3 \cap \mathbb{Z}_3) \times \ldots \times \underset{p\text{-th position}}{(\varnothing)} \times \ldots, \end{aligned}$$

in $A_{\mathbb{Q}}$, by the chain property in (2.2), which will gives

$$\varphi_{p,j}(\alpha_Y) = 0, \text{ by } (5.1).$$

(Also, see (5.5) below.) It shows that, whenever $j < 0$ in $\mathbb{Z}$, and $p \notin P_Y$, one can get vanishing free-moments.

Also, suppose that $j < 0$ in $\mathbb{Z}$, and say $p = p_1 \in P_Y$ in $\mathcal{P}$, for convenience. Then

$$Y \cap B_j^p = Y \cap B_j^{p_1} = (S_2 \cap \mathbb{Z}_2) \times \ldots \times \underset{p_1\text{-th position}}{\left(\partial_{k_{p_1}}^{p_1} \cap \partial_j^{p_1}\right)} \times \ldots,$$

and hence,

$$\varphi_{p,j}(\alpha_Y) = \delta_{k_{p_1},j}\varphi_{p,j}(\alpha_Y) = \begin{cases} \varphi_{p,j}(\alpha_Y) & \text{if } k_{p_1} = j < 0, \\ 0 & \text{if } k_{p_1} \geq 0, \end{cases}$$

by (5.1). (Also, see (5.5) below.)

So, for most of arbitrary choices of Y, the quantities $\varphi(\alpha_Y)$ are vanishing.

Therefore, to avoid-or-overcome the above two vanishing cases, we determine our system (5.1) of linear functionals

$$\{\varphi_{p,j} : p \in \mathcal{P}, j \in \mathbb{N}_0\},$$

by taking j from $\mathbb{N}_0$ in $\mathbb{Z}$.

Remark also that, even though we take a system

$$\{\varphi_{p,j} : p \in \mathcal{P}, \text{“}j \in \mathbb{Z}\text{”}\}$$

of linear functionals $\varphi_{p,j}$ in the sense of (5.1), one can get similar results like our main results of this Section (containing vanishing cases). However, we need to polish lots of vanishing cases. Thus, our system of linear functionals would be chosen by (5.1) for convenience.

Note that all vectors h of $H_{\mathcal{P}}$ have their expressions,

$$h = \sum_{Y \in \sigma(A_{\mathbb{Q}})} t_Y \chi_Y, \text{ with } t_Y \in \mathbb{C},$$

where $\sum$ is a finite, or an infinite (limit of finite) sum(s) under the Hilbert-space topology induced by the metric (4.8).

Note also that every operator T of $M_{\mathcal{P}}$ has its expression,

$$T = \sum_{Y \in \sigma(A_{\mathbb{Q}})} s_Y \alpha_Y, \text{ with } s_Y \in \mathbb{C},$$

where $\sum$ is a finite, or an infinite (limit of finite) sum(s) under the C^*-topology for $M_{\mathcal{P}}$, and where α_Y are in the sense of Notation 4.3.

Therefore, the linear functionals $\varphi_{p,j}$ of (5.1) are well-defined on $M_{\mathcal{P}}$, and hence, one can get the corresponding C^*-probability spaces

$$M_{\mathcal{P}}^{p,j} \overset{denote}{=} (M_{\mathcal{P}},\ \varphi_{p,j}), \tag{5.2}$$

for all $p \in \mathcal{P}, j \in \mathbb{N}_0$.

Definition 5.2. Let $M_{\mathcal{P}}^{p,j} = (M_{\mathcal{P}}, \varphi_{p,j})$ be a C^*-probability space (5.2), for $p \in \mathcal{P}$, $j \in \mathbb{N}_0$. Then we call $M_{\mathcal{P}}^{p,j}$, the (p,j)(-finite)-Adelic C^*-probability space of the finite-Adelic C^*-algebra $M_{\mathcal{P}}$.

In the rest of this section, let us fix $p \in \mathcal{P}$ and $j \in \mathbb{N}_0$, and the corresponding (p, j)-Adelic C^*-probability space $M_{\mathcal{P}}^{p,j}$ of (5.2).

Consider first that, let $\alpha_Y = \alpha_{\chi_Y} \in M_{\mathcal{P}}^{p,j}$, for $Y \in \sigma(A_{\mathbb{Q}})$, satisfying

$$Y = \prod_{q \in \mathcal{P}} S_q, \text{ with } S_q \in \sigma(\mathbb{Q}_p), \tag{5.3}$$

where

$$S_q = \begin{cases} S_q \neq \mathbb{Z}_q & \text{if } q \in P_Y, \\ \mathbb{Z}_q & \text{if } q \notin P_Y, \end{cases}$$

where

$$P_Y = \{q \in \mathcal{P} : S_q \neq \mathbb{Z}_q\} \text{ in } \mathcal{P}.$$

Then we have

$$\varphi_{p,j}(\alpha_Y) = \left[\alpha_Y(\chi_{B_j^p}),\ \chi_{B_j^p}\right]$$

where B_j^p is in the sense of (5.1) in $\sigma(A_{\mathbb{Q}})$

$$\begin{aligned}
&= \left[\chi_Y \chi_{B_j^p},\ \chi_{B_j^p}\right] = \left[\chi_{Y \cap B_j^p},\ \chi_{B_j^p}\right] \\
&= \int_{A_{\mathbb{Q}}} \chi_{Y \cap B_j^p} \chi_{B_j^p}^* d\mu = \int_{A_{\mathbb{Q}}} \chi_{Y \cap B_j^p} \chi_{B_j^p} d\mu \\
&= \int_{A_{\mathbb{Q}}} \chi_{Y \cap B_j^p \cap B_j^p} d\mu = \int_{A_{\mathbb{Q}}} \chi_{Y \cap B_j^p} d\mu \\
&= \mu\left(Y \cap B_j^p\right) \\
&= \begin{cases} \left(\prod_{q \in P_Y} \mu_q\left(S_q \cap \mathbb{Z}_q\right)\right)\left(\mu_p\left(\partial_j^p\right)\right) & \text{if } p \notin P_Y, \\ \left(\prod_{q \in P_Y \setminus \{p\}} \mu_q\left(S_q \cap \mathbb{Z}_q\right)\right)\left(\mu_p\left(S_p \cap \partial_j^p\right)\right) & \text{if } p \in P_Y. \end{cases}
\end{aligned} \tag{5.4}$$

Note again that the formula (5.4) is obtained because we take j in $\mathbb{N}_0$ (not in $\mathbb{Z}$: if $j < 0$, then the above formula (5.4) vanishes). By (5.4), we obtain the following result.

Theorem 5.3. *Let α_Y be a free random variable in the (p, j)-Adelic C^*-probability space $M_{\mathcal{P}}^{p,j}$, where $Y \in \sigma(A_{\mathbb{Q}})$ is in the sense of (5.3). Then*

$$\varphi\left(\alpha_Y^n\right) = \left(\prod_{q \in (P_Y \cup \{p\}) \setminus \{p\}} \mu_q\left(S_q \cap \mathbb{Z}_q\right)\right)\left(\mu_p\left(S_p \cap \partial_j^p\right)\right), \tag{5.5}$$

for all $n \in \mathbb{N}$.

Proof. Let α_Y be as above in $M_{\mathcal{P}}^{p,j}$. Then

$$\alpha_Y^n = \left(\alpha_{\chi_Y}\right)^n = \alpha_{\chi_Y^n} = \alpha_{\chi_Y} = \alpha_Y,$$

in $M_{\mathcal{P}}^{p,j}$, for all $n \in \mathbb{N}$. So,

$$\varphi_{p,j}\left(\alpha_Y^n\right) = \varphi\left(\alpha_Y\right), \text{ for all } n \in \mathbb{N}.$$

The quantity $\varphi(\alpha_Y)$ is obtained in (5.4), re-expressed simply by (5.5). Note that the formula (5.5), indeed, implies the two cases of (5.4) altogether. □

Now, let Y be in the sense of (5.3), with specific condition as follows;

$$Y = \prod_{q \in \mathcal{P}} S_q, \text{ with } S_q \in \sigma(\mathbb{Q}_p),$$

where

$$S_q = \begin{cases} \partial_{k_q}^q & \text{if } q \in P_Y, \\ \mathbb{Z}_q & \text{if } q \notin P_Y, \end{cases} \tag{5.6}$$

for all $q \in \mathcal{P}$, where $k_q \in \mathbb{Z}$ for $q \in P_Y$, and

$$P_Y = \{p_1, \ldots, p_N\} \text{ in } \mathcal{P}, \text{ for some } N \in \mathbb{N}.$$

If Y is in the sense of (5.6), then the corresponding free random variable α_Y of the (p, j)-Adelic C^*-probability space $M_{\mathcal{P}}^{p,j}$ satisfies that

$$\varphi\left(\alpha_Y^n\right) = \left(\prod_{q \in (P_Y \cup \{p\}) \setminus \{p\}} \mu_q\left(\partial_{k_q}^q \cap \mathbb{Z}_q\right)\right)\left(\mu_p\left(S_p \cap \partial_j^p\right)\right)$$

by (5.5)

$$= \begin{cases} \left(\prod_{q \in P_Y} \mu_q\left(\partial_{k_q}^q \cap \mathbb{Z}_q\right)\right)\left(\mu_p\left(\partial_j^p\right)\right) & \text{if } p \notin P_Y, \\ \left(\prod_{q \in P_Y \setminus \{p\}} \mu_q\left(\partial_{k_q}^q \cap \mathbb{Z}_q\right)\right)\left(\mu_p\left(\partial_{k_p}^p \cap \partial_j^p\right)\right) & \text{if } p \in P_Y, \end{cases}$$

$$= \begin{cases} \left(\prod_{q \in P_Y} \mu_q\left(\partial_{k_q}^q \cap \mathbb{Z}_q\right)\right)\left(\mu_p\left(\partial_j^p\right)\right) & \text{if } p \notin P_Y, \\ \delta_{j,k_p}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right)\left(\prod_{q \in P_Y \setminus \{p\}} \mu_q\left(\partial_{k_q}^q \cap \mathbb{Z}_q\right)\right) & \text{if } p \in P_Y, \end{cases} \tag{5.7}$$

for all $n \in \mathbb{N}$, where δ means the *Kronecker delta.*

Therefore, one obtains the following corollary of (5.5), with help of (5.7).

Corollary 5.4. *Let Y be in the sense of* (5.6) *in* $\sigma(A_{\mathbb{Q}})$, *and let α_Y be the corresponding free random variable of the (p, j)-Adelic C^*-probability spae $M_{\mathcal{P}}^{p,j}$. Then*

$$\varphi_{p,j}\left(\alpha_Y^n\right) = \delta_{j,Y}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right)\left(\prod_{q \in P_Y \setminus \{p\}} \mu_q\left(\partial_{k_q}^q \cap \mathbb{Z}_q\right)\right), \tag{5.8}$$

for all $n \in \mathbb{N}$, *where*

$$\delta_{j,Y} = \begin{cases} \delta_{j,k_p} & \text{if } p \in P_Y, \\ 1 & \text{otherwise,} \end{cases}$$

where P_Y *is in the sense of* (5.6).

Proof. The free-moment formula (5.8) holds by (5.5) and (5.7). If we simplify the expression (5.7), then the formula (5.8) is obtained. □

Note that the operator α_Y of the above corollary is nothing but an operator induced by a *boundary-product element* χ_Y of the finite-Adelic probability space $(\mathcal{M}_{\mathcal{P}}, \varphi)$, and hence, they provide building blocks of free distributions of all operators in $M_{\mathcal{P}}$ from (5.8). So, as in Section 3, we focus on studying free-distributional data of these operators α_Y for investigating free distributions of all operators of $M_{\mathcal{P}}$.

Definition 5.5. Let α_Y be the operator of the finite-Adelic C^*-algebra $M_{\mathcal{P}}$, generated by the μ-measurable subset Y of (5.6). Then we call such an operator α_Y a boundary-product operator of $M_{\mathcal{P}}$.

As we discussed above, in the rest of this paper, we focus on studying free-distributional data of certain operators of $M_{\mathcal{P}}$, generated by boundary-product operators α_Y's in $M_{\mathcal{P}}^{p,j}$, for all $p \in \mathcal{P}$, and $j \in \mathbb{N}_0$, where Y are in the sense of (5.6) in $\sigma(A_{\mathbb{Q}})$.

Note that, if Y is in the sense of (5.6) and if

$$k_p \in \mathbb{N}_0 \text{ in } \mathbb{Z}, \text{ for all } p \in P_Y, \tag{5.9}$$

then it is regarded as

$$Y = \bigcap_{q \in P_Y} B_{k_q}^q \text{ in } \sigma(A_{\mathbb{Q}}), \tag{5.10}$$

where $B_{k_q}^q$ are the μ-measurable subsets of $A_{\mathbb{Q}}$ in the sense of (5.1), for $q \in P_Y$ and for now $k_q \in \mathbb{Z}$, where P_Y is the subset (5.6) of $\mathcal{P}$. Note that the above set-equality (5.10) holds only if the condition (5.9) of Y is satisfied.

Therefore, the corresponding boundary-product operator α_Y is understood as

$$\alpha_Y = \alpha_{\bigcap\limits_{q \in P_Y} B_{k_q}^q} = \prod_{q \in P_Y} \alpha_{B_{k_q}^q} \tag{5.11}$$

in $M_{\mathcal{P}}^{p,j}$, under (5.10). So, one can get that

$$\varphi_{p,j}(\alpha_Y) = \varphi_{p,j}\left(\prod_{p \in P_Y} \alpha_{B_{k_q}^q}\right)$$

by (5.10)

$$= \left[\left(\prod_{p \in P_Y} \alpha_{B_{k_q}^q}\right)\left(\chi_{B_j^p}\right),\ \chi_{B_j^p}\right]$$

$$= \left[\chi_{\underset{q \in P_Y \cap \{p\}}{\cap} B_{k_q}^q},\ \chi_{B_j^p}\right]$$

by identifying $k_p = j$ in $\mathbb{N}_0$

$$= \delta_{j,Y}\left(\prod_{q \in P_Y \cup \{p\}} \mu_q\left(\partial_{k_q}^q\right)\right) = \delta_{j,Y}\left(\prod_{q \in P_Y \cup \{p\}}\left(\frac{1}{q^{k_q}} - \frac{1}{q^{k_q+1}}\right)\right), \tag{5.12}$$

where $\delta_{j,Y}$ is in the sense of (5.8). Therefore, from a different approach from (5.9), we obtain the following special case of (5.8).

Corollary 5.6. *Let Y be in the sense of* (5.6) *with additional condition* (5.9) *in $\sigma(A_{\mathbb{Q}})$, and let α_Y be the corresponding boundary-product operator in the (p,j)-Adelic C^*-probability space $M_{\mathcal{P}}^{p,j}$, for $p \in \mathcal{P}$, and $j \in \mathbb{N}_0$. Then*

$$\alpha_Y = \prod_{q \in P_Y} \alpha_{B_{k_q}^q} \text{ in } M_{\mathcal{P}},$$

and

$$\varphi_{p,j}\left(\alpha_Y^n\right) = \delta_{j,Y}\left(\prod_{q \in P_Y \cup \{p\}}\left(\frac{1}{q^{k_q}} - \frac{1}{q^{k_q+1}}\right)\right), \tag{5.13}$$

with identification: $k_p = j$ in $\mathbb{Z}$, for all $n \in \mathbb{N}$, where $\delta_{j,Y}$ is in the sense of (5.8).

Proof. The operator-identity in (5.13) is shown by (5.11), and the free-moment formula in (5.13) is proven by (5.8) and (5.12), since $\alpha_Y^n = \alpha_Y$ in $M_{\mathcal{P}}$, for all $n \in \mathbb{N}$, for all $p \in \mathcal{P}, j \in \mathbb{N}_0$. $\square$

Now, let Y and let P_Y be in the sense of (5.6) (not necessarily with (5.9)). Then P_Y is partitioned by

$$P_Y = P_Y^+ \sqcup P_Y^- \text{ in } \mathcal{P}, \tag{5.14}$$

where

$$P_Y^+ = \{q \in P_Y : k_q \geq 0 \text{ in } \mathbb{Z}\},$$

and

$$P_Y^- = \{q \in P_Y : k_q < 0 \text{ in } \mathbb{Z}\}.$$

Then the formula (5.8) can be refined as follows with help of (5.13).

Theorem 5.7. *Let Y be in the sense of* (5.6)*, inducing a finite subset $P_Y = P_Y^+ \sqcup P_Y^-$ of $\mathcal{P}$, as in* (5.14)*. If $\alpha_Y \in M_{\mathcal{P}}^{p,j}$, for $p \in \mathcal{P}, j \in \mathbb{N}_0$, then*

$$\varphi_{p,j}\left(\alpha_Y^n\right) = \begin{cases} \delta_{j,Y}\left(\prod_{q \in P_Y}\left(\frac{1}{q^{k_q}} - \frac{1}{q^{k_q+1}}\right)\right) & \text{if } P_Y^- = \varnothing, \\ 0 & \text{if } P_Y^- \neq \varnothing, \end{cases} \tag{5.15}$$

for all $n \in \mathbb{N}$, where $\varnothing$ means the empty set.

Proof. Suppose α_Y be given as above in $M_{\mathcal{P}}^{p,j}$. Assume first that

$$P_Y^- \neq \varnothing \text{ in } P_Y \subset \mathcal{P},$$

and let

$$P_Y^- = \{q_1, \ldots, q_t\}, \text{ for some } t \leq N \text{ in } \mathbb{N},$$

i.e., $k_{q_s} < 0$ in $\mathbb{Z}$, for all $s = 1, \ldots, t$. Then

$$\begin{aligned} Y \cap B_j^p &= (S_2 \cap \mathbb{Z}_2) \times \ldots \times \underbrace{\left(\partial_{k_{q_1}}^{q_1} \cap \mathbb{Z}_{q_1}\right)}_{q_1\text{-th position}} \times \ldots \times \underbrace{\left(\partial_{k_{q_t}}^{q_t} \cap \mathbb{Z}_{q_t}\right)}_{q_t\text{-th position}} \times \ldots \\ &= (S_2 \cap \mathbb{Z}_2) \times \ldots \times (\varnothing) \times \ldots \times (\varnothing) \times \ldots, \end{aligned}$$

by the chain property of $\mathbb{Q}_{q_s}$ in (2.2), for $s = 1, \ldots, t$, in $A_{\mathbb{Q}}$. Thus,

$$\varphi_{p,j}\left(\alpha_Y^n\right) = \varphi\left(\alpha_Y\right) = 0,$$

by (5.8), for all $n \in \mathbb{N}$.

Assume now that $P_Y^- = \varnothing$, equivalently, suppose $P_Y = P_Y^+$ in $\mathcal{P}$. Then, by (5.13), one obtains that

$$\varphi_{p,j}\left(\alpha_Y^n\right) = \delta_{j,Y}\left(\prod_{q \in P_Y^+ \cup \{p\}}\left(\frac{1}{q^{k_q}} - \frac{1}{q^{k_q+1}}\right)\right)$$

for all $n \in \mathbb{N}$, where

$$\delta_{j,Y} = \begin{cases} \delta_{j,k_p} & \text{if } p \in P_Y^+ = P_Y, \\ 1 & \text{otherwise,} \end{cases}$$

by (5.13).

Therefore, the refined free-moment formula (5.15) of (5.8) holds. □

The above free-distributional data (5.15) shows that, all μ-measurable subsets Y inducing non-empty subset P_Y^- generate non-zero operators α_Y in our (p,j)-Adelic C^*-probability spaces $M_{\mathcal{P}}^{p,j}$, having vanishing free distributions, for all $p \in \mathcal{P}$ and $j \in \mathbb{N}_0$.

Assumption and Notation 5.8 (in short, AN 5.8 from below)**.** Let Y be in the sense of (5.6) satisfying (5.14), and let α_Y be the corresponding boundary-product operator in the finite-Adelic C^*-algebra $M_{\mathcal{P}}$. In the rest of this paper, we automatically assume

$$P_Y^- = \varnothing, \text{ equivalently, } P_Y = P_Y^+,$$

i.e., from below,

$$P_Y = \{q \in \mathcal{P} : k_q \geq 0 \text{ in } \mathbb{Z}\} = P_Y^+,$$

to avoid the vanishing cases in (5.15).

To avoid confusion, we will say such boundary-product operators α_Y are $(+)$-*boundary*$($-*product*$)$ *operators of* $M_{\mathcal{P}}$. The notation is reasonable because such $(+)$-boundary operators α_Y are induced by $(+)$-boundary elements χ_Y of $\mathcal{M}_{\mathcal{P}}$.$\square$

From below, all boundary-product operators would be $(+)$-boundary operators of AN 5.8 in $M_{\mathcal{P}}$. Again, notice that all boundary-product operators α_Y, which are not $(+)$-boundary operators in $M_{\mathcal{P}}^{p,j}$, have vanishing free distributions, for all $p \in \mathcal{P}$, and $j \in \mathbb{N}_0$, by (5.15). So, we focus on studying free-distributional data of $(+)$-boundary operators in (p,j)-Adelic C^*-probability spaces $M_{\mathcal{P}}^{p,j}$, for all $p \in \mathcal{P}, j \in \mathbb{N}_0$.

Theorem 5.9. *Let α_Y be a $(+)$-boundary operator of AN 5.8 in the finite-Adelic C^*-algebra $M_{\mathcal{P}}$. Let us understand α_Y as a free random variable in the (p,j)-Adelic C^*-probability space $M_{\mathcal{P}}^{p,j}$, for $p \in \mathcal{P}$, and $j \in \mathbb{N}_0$. Then there exist*

$$n_Y = \prod_{q \in P_Y^+ \cup \{p\}} q^{k_q}, \text{ with identification: } k_p = j \text{ in } \mathbb{N}_0,$$

in $\mathbb{N}$, *such that*

$$\varphi_{p,j}\left(\alpha_Y^n\right) = \frac{\delta_{j,Y}}{n_Y n_{p,j}} \phi(n_{p,j}), \text{ for all } n \in \mathbb{N}, \tag{5.16}$$

where

$$n_{p,j} = \prod_{q \in P_Y^+ \cup \{p\}} q \text{ in } \mathbb{N},$$

where $\delta_{j,Y}$ is in the sense of (5.8)*, and ϕ is the Euler totient function.*

Proof. Recall that, for a fixed $p \in \mathcal{P}, j \in \mathbb{N}_0$, if Y is a μ-measurable set of $A_{\mathbb{Q}}$, satisfying both (5.6) and (5.9), then the corresponding operator α_Y forms a $(+)$-boundary operator of AN 5.8 in $M_{\mathcal{P}}^{p,j}$, and it satisfies

$$\varphi_{p,j}(\alpha_Y) = \delta_{j,Y}\left(\prod_{q \in P_Y \cup \{p\}} \left(\frac{1}{q^{k_q}} - \frac{1}{q^{k_q+1}}\right)\right), \tag{5.17}$$

with identification: $k_p = j$ in $\mathbb{Z}$, since $P_Y = P_Y^+$, where

$$\delta_{j,Y} = \begin{cases} \delta_{j,k_p} & \text{if } p \in P_Y = P_Y^+, \\ 1 & \text{otherwise,} \end{cases}$$

by (5.8) and (5.13).

Then, one can re-write the formula (5.17) as follows:

$$\begin{aligned}\varphi_{p,j}(\alpha_Y) &= \delta_{j,Y}\left(\prod_{q\in P_{Y,p}}\left(\frac{1}{q^{k_q}}-\frac{1}{q^{k_q+1}}\right)\right)\\ &= \delta_{j,Y}\left(\prod_{q\in P_{Y,p}}\frac{q^{k_q}}{q^{k_q+1}}\left(1-\frac{1}{q}\right)\right)\\ &= \delta_{j,Y}\left(\prod_{q\in P_{Y,p}}\left(\frac{1}{q^{k_q}}\right)\left(\frac{1}{q}\right)\right)\left(\prod_{q\in P_{Y,p}^{+}} q\left(1-\frac{1}{q}\right)\right)\\ &= \delta_{j,Y}\left(\frac{1}{n_Y}\right)\left(\frac{1}{n_{p,j}}\right)\phi(n_{p,j}),\end{aligned}$$

where

$$n_Y = \prod_{q\in P_{Y,p}} q^{k_q}, \quad n_{p,j} = \prod_{q\in P_{Y,p}} q,$$

in $\mathbb{N}$, and hence, it goes to

$$= \delta_{j,Y}\left(\frac{1}{n_Y n_{p,j}}\right)\phi(n_{p,j}), \tag{5.18}$$

for all $n \in \mathbb{N}$. □

The above two theorems illustrate relations between our C^*-probabilistic structures, and number-theoretic information by (5.15) and (5.16). Also, they show a conncetion between the $*$-probabilistic data (3.21), and the C^*-probabilistic data (5.18), whenever $P_Y = P_Y^+$ in $\mathcal{P}$.

In the rest parts of this paper, we study (+)-boundary operators α_Y of AN 5.8 in the finite-Adelic C^*-algebra $M_{\mathcal{P}}$, and the free distributions of certain operators, generated by these (+)-boundary operators, in (p,j)-Adelic C^*-probability spaces $M_{\mathcal{P}}^{p,j}$ for all $p \in \mathcal{P}$ and $j \in \mathbb{Z}$.

6. DISTRIBUTIONS INDUCED BY (+)-BOUNDARY OPERATORS

The main purposes of this section is to consider free-distributional data of (+)-boundary operators which provide the building blocks of operators in $M_{\mathcal{P}}$ having possible non-vanishing free distributions.

Let $M_{\mathcal{P}}$ be the finite-Adelic C^*-algebra generated by the finite-Adelic probability space $(\mathcal{M}_{\mathcal{P}}, \varphi)$ under the finite-Adelic representation $(H_{\mathcal{P}},\ \alpha)$, and let

$$M_{\mathcal{P}}^{p,j} = (M_{\mathcal{P}}, \varphi_{p,j})$$

be the (p,j)-Adelic C^*-probability spaces (5.2), for all $p \in \mathcal{P}, j \in \mathbb{N}_0$.

Let $Y = \prod_{q \in \mathcal{P}} S_q = \underset{q \in P_Y}{\cap} B_{k_q}^q$ be μ-measurable subsets of the finite Adele ring $A_{\mathbb{Q}}$ in $\sigma(\mathbb{Q}_q)$, where

$$B_{k_q}^q = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \ldots \times \underset{q\text{-th}}{\partial_{k_q}^q} \times \cdots$$

in $A_{\mathbb{Q}}$, and

$$P_Y = \{q \in \mathcal{P} : S_q = \partial_{k_q}^q\}$$

is a finite subset of $\mathcal{P}$. Equivalently, the subsets Y of $A_{\mathbb{Q}}$ are in the sense of (5.6). Moreover, assume that Y satisfies the condition (5.9) as in AN 5.8, too, i.e.,

$$P_Y = P_Y^+ \Longleftrightarrow P_Y^- = \varnothing \text{ in } \mathcal{P}.$$

Then the corresponding (+)-boundary opeartors α_Y are well-determined in $M_{\mathcal{P}}$.

Recall that if α_Y is a (+)-boundary operator in the sense of AN 5.8, then, as a free random variable in a (p, j)-Adelic C^*-probability space $M_{\mathcal{P}}^{p,j}$, one obtains

$$\varphi(\alpha_Y^n) = \frac{\delta_{j,Y}}{n_Y n_{p,j}} \phi(n_{p,j}) \text{ for all } n \in \mathbb{N}, \tag{6.1}$$

by (5.15), (5.16) and (5.18), where ϕ is the Euler totient function (3.14) and where

$$\delta_{j,Y} = \begin{cases} \delta_{j,k_q} & \text{if } p \in P_Y = P_Y^+, \\ 1 & \text{otherwise,} \end{cases}$$

$$n_Y = \prod_{q \in P_{Y,p}} q^{k_q} \text{ and } n_{p,j} = \prod_{q \in P_{Y,p}} q \text{ in } \mathbb{N},$$

where

$$P_{Y,p} = P_Y \cup \{p\} = P_{Y,p}^+ \cup \{p\},$$

which is a finite subset of $\mathcal{P}$.

Now, let $Y_1, \ldots, Y_N \in \sigma(A_{\mathbb{Q}})$ be in the sense of (5.6) with (5.9), and let α_{Y_l} be the corresponding (+)-boundary operators in $M_{\mathcal{P}}$, for all $l = 1, \ldots, N$, for $N \in \mathbb{N}$. Define a new operator $T_{1,\ldots,N} \in M_{\mathcal{P}}$ by

$$T_{1,\ldots,N} = \prod_{l=1}^{N} \alpha_{Y_l} \in M_{\mathcal{P}}. \tag{6.2}$$

By the very construction (6.2) of $T_{1,\ldots,N}$, one can get that

$$T_{1,\ldots,N} = \prod_{l=1}^{N} \alpha_{Y_l} = \alpha_{\overset{N}{\underset{l=1}{\cap}} Y_l} = \prod_{q \in \overset{n}{\underset{l=1}{\bigcup}} P_{Y_l}} \alpha_{B_{k_q}^q}, \tag{6.3}$$

in $M_{\mathcal{P}}$, "under (5.9)". (Remark that, without the condition (5.9), the relation (6.3) does not hold, in general, because of the vanishing cases.)

The observation (6.3) shows that there exists a μ-measurable subset $Y_{1,\ldots,N}$ of $A_{\mathbb{Q}}$,

$$Y_{1,\ldots,N} = \prod_{q \in P_{Y_{1,\ldots,N}}} B^q_{k_q} \text{ in } \sigma(A_{\mathbb{Q}}), \tag{6.4}$$

such that

$$T_{1,\ldots,N} = \alpha_{Y_{1,\ldots,N}} \text{ in } M_{\mathcal{P}}, \tag{6.5}$$

by (6.4), where

$$P_{Y_{1,\ldots,N}} = \bigcup_{l=1}^{n} P_{Y_l} = \bigcup_{l=1}^{n} P^+_{Y_l} = P^+_{Y_{1,\ldots,N}}$$

is a finite subset of $\mathcal{P}$, where P_{Y_l} are in the sense of AN 5.8, for all $l = 1, \ldots, N$.

Therefore, the μ-measurable subset $Y_{1,\ldots,N}$ of (6.4) also satisfies AN 5.8, and hence, the corresponding operator $T_{1,\ldots,N}$ of (6.2) forms a new (+)-boundary operator $\alpha_{Y_{1,\ldots,N}}$ in $M_{\mathcal{P}}$, by (6.5). It shows that the products of (+)-boundary operators become (+)-boundary operators in $M_{\mathcal{P}}$.

Lemma 6.1. *Let $T_{1,\ldots,N}$ be the operator product* (6.2) *of* (+)*-boundary operators $\alpha_{Y_1}, \ldots, \alpha_{Y_N}$ of AN 5.8 in $M_{\mathcal{P}}$, for some $N \in \mathbb{N}$. Then there exists*

$$Y_{1,\ldots,N} = \bigcap_{l=1}^{N} Y_l \in \sigma(A_{\mathbb{Q}}),$$

such that

$$T_{1,\ldots,N} = \alpha_{Y_{1,\ldots,N}} \in M_{\mathcal{P}}, \tag{6.6}$$

where $\alpha_{Y_{1,\ldots,N}}$ is a new (+)*-boundary operator in $M_{\mathcal{P}}$.*

Proof. The existence of the μ-measurable subset $Y_{1,\ldots,N}$ is guaranteed by (6.3) and (6.4), and the operator equality (6.6) is proven by (6.5). $\square$

Since operator products of (+)-boundary operators are (+)-boundary operators in $M_{\mathcal{P}}$, by (6.6), we obtain the following free-probabilistic information.

Theorem 6.2. *Let $T_{1,\ldots,N}$ be an operator* (6.2) *in $M_{\mathcal{P}}$. As a free random variable in a (p,j)-Adelic C^*-probability space $M^{p,j}_{\mathcal{P}}$, for $p \in \mathcal{P}, j \in \mathbb{N}_0$, we have that*

$$\begin{aligned} \varphi_{p,j}\left((T_{1,\ldots,N})^n\right) &= \delta_{j,Y_{1,\ldots,N}} \left(\prod_{q \in P_{Y_{1,\ldots,N}:p}} \left(\frac{1}{q^{k_q}} - \frac{1}{q^{k_q+1}} \right) \right) \\ &= \frac{\delta_{j,Y_{1,\ldots,N}}}{n_{Y_{1,\ldots,N}} n_{p,j}} \phi\left(n_{p,j}\right), \end{aligned} \tag{6.7}$$

for all $n \in \mathbb{N}$, where

$$P_{Y_{1,\ldots,N}:p} = P_{Y_{1,\ldots,N}} \cup \{p\} = P^+_{Y_{1,\ldots,N}} \cup \{p\},$$

and

$$P_{Y_{1,\ldots,N}} = \{q \in \mathcal{P} : S_q = \partial^q_{k_q}\} = P^+_{Y_{1,\ldots,N}},$$

are the finite subsets of $\mathcal{P}$, *whenever*

$$Y_{1,\ldots,N} = \prod_{q \in \mathcal{P}} S_q \in \sigma(A_{\mathbb{Q}}), \text{ with } S_q \in \sigma(\mathbb{Q}_q),$$

and where

$$\delta_{j,Y_{1,\ldots,N}} = \begin{cases} \delta_{j,k_q} & \text{if } p \in P_{Y_{1,\ldots,N}}, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$n_{Y_{1,\ldots,N}} = \prod_{q \in P_{Y_{1,\ldots,N}:p}} q^{k_q}, \text{ and } n_{p,j} = \prod_{q \in P_{Y_{1,\ldots,N}::p}} q,$$

in $\mathbb{N}$.

Proof. By (6.6), there exists $Y_{1,\ldots,N} = \overset{N}{\underset{l=1}{\cap}} Y_l \in \sigma(A_{\mathbb{Q}})$, such that $T_{1,\ldots,N} = \alpha_{Y_{1,\ldots,N}}$, as a new (+)-boundary operator in $M^{p,j}_{\mathcal{P}}$. Since it is a new (+)-boundary operator, one has

$$(T_{1,\ldots,N})^n = (\alpha_{Y_{1,\ldots,N}})^n = \alpha_{Y_{1,\ldots,N}} = T_{1,\ldots,N},$$

for all $n \in \mathbb{N}$. So, one obtains that

$$\varphi_{p,j}\left((T_{1,\ldots,N})^n\right) = \varphi_{p,j}\left(T_{1,\ldots,N}\right) = \varphi_{p,j}\left(\alpha_{Y_{1,\ldots,N}}\right)$$
$$= \delta_{j,Y_{1,\ldots,N}} \left(\prod_{q \in P_{Y_{1,\ldots,N}:p}} \left(\frac{1}{q^{k_q}} - \frac{1}{q^{k_q+1}} \right) \right)$$

by (5.13), where

$$P_{Y_{1,\ldots,N}:p} = P_{Y_{1,\ldots,N}} \cup \{p\},$$

and

$$P_{Y_{1,\ldots,N}} = \{q \in \mathcal{P} : S_q = B^q_{k_q}\} = P^+_{Y_{1,\ldots,N}}$$

is a finite subset of $\mathcal{P}$, whenever

$$Y_{1,\ldots,N} = \prod_{q \in \mathcal{P}} S_q, \text{ with } S_q \in \sigma(\mathbb{Q}_q),$$

and hence, it goes to

$$= \frac{\delta_{j,Y_{1,\ldots,N}}}{n_{Y_{1,\ldots,N}} n_{p,j}} \phi(n_{p,j}),$$

where

$$n_{Y_{1,\ldots,N}} = \prod_{q \in P_{Y_{1,\ldots,N}:p}} q^{k_q}, \text{ and } n_{p,j} = \prod_{q \in P_{Y_{1,\ldots,N}:p}} q$$

in $\mathbb{N}$, by (5.18), where ϕ is the Euler totient function, for all $n \in \mathbb{N}$. Therefore, the free-moment formula (6.7) holds. $\square$

Now, let B_k^q be in the sense of (5.1) in $\sigma(A_{\mathbb{Q}})$, for $q \in \mathcal{P}$, and "$k \in \mathbb{N}_0$", i.e.,

$$B_k^q = \prod_{s \in \mathcal{P}} S_s,$$

with

$$S_s = \begin{cases} \partial_k^q & \text{if } s = q, \\ \mathbb{Z}_s & \text{otherwise,} \end{cases} \tag{6.8}$$

in $\mathbb{Q}_s$, for $s \in \mathcal{P}$.

Then it provides the corresponding operator $\alpha_{B_k^q}$ in the finite-Adelic C^*-algebra $M_{\mathcal{P}}$, as a $(+)$-boundary operator in the sense of AN 5.8, because $k \geq 0$ in $\mathbb{Z}$, i.e., it induces

$$P_{B_k^q} = \{q\} = P_{(B_k^q)^+}, \text{ with } k_q = k \geq 0.$$

Let $B_{j_1}^{p_1}, \ldots, B_{j_N}^{p_N}$ be the μ-measurable subsets (6.8) in the finite Adele ring $A_{\mathbb{Q}}$, where $p_1, \ldots, p_N$ are "mutually distinct" from each other in $\mathcal{P}$, and $j_1, \ldots, j_N \in \mathbb{N}_0$ (which are not necessarily distinct), for $N \in \mathbb{N}$. So, these sets automatically satisfy AN 5.8. Now, let

$$\alpha_{p_l, j_l} = \alpha_{B_{j_l}^{p_l}} \in M_{\mathcal{P}}, \tag{6.9}$$

be the corresponding $(+)$-boundary operators, for all $l = 1, \ldots, N$. Construct now a new operator

$$S = \sum_{l=1}^{N} \alpha_{p_l, j_l} \text{ in } M_{\mathcal{P}}, \tag{6.10}$$

where α_{p_l, j_l} are in the sense of (6.9), for all $l = 1, \ldots, N$.

Observe that, if S is an operator (6.10) in $M_{\mathcal{P}}$, then

$$\begin{aligned} S^n &= \sum_{(l_1, \ldots, l_n) \in \{1, \ldots, N\}^n} \left(\prod_{s=1}^{n} \alpha_{p_{l_s}, j_{l_s}} \right) \\ &= \sum_{(l_1, \ldots, l_n) \in \{1, \ldots, N\}^n} \alpha_{Y_{l_1, \ldots, l_n}}, \end{aligned} \tag{6.11}$$

where

$$Y_{l_1, \ldots, l_n} = \bigcap_{s=1}^{n} B_{j_{l_s}}^{p_{l_s}} \in \sigma(A_{\mathbb{Q}}),$$

for all $(l_1, \ldots, l_n) \in \{1, \ldots, N\}^n$, for all $n \in \mathbb{N}$.

Remark that the summands $\alpha_{Y_{l_1, \ldots, l_n}}$ of S^n in (6.11) form $(+)$-boundary operators, for all $(l_1, \ldots, l_n) \in \{1, \ldots, N\}^n$, for all $n \in \mathbb{N}$. Thus, we obtain the following free-distributional data of the operators S of (6.10).

Corollary 6.3. *Let S be an operator (6.10) in $M_{\mathcal{P}}$. Then, as a free random variable in the (p, j)-Adelic C^*-probability space $M_{\mathcal{P}}^{p,j}$, for $p \in \mathcal{P}$, and $j \in \mathbb{N}_0$, it satisfies that*

$$\varphi_{p,j}(S^n) = \sum_{(l_1, \ldots, l_n) \in \{1, \ldots, N\}^n} \sum \varphi_{p,j}\left(\alpha_{Y_{l_1, \ldots, l_n}}\right), \tag{6.12}$$

where $\alpha_{Y_{l_1,\ldots,l_n}}$ *are in the sense of* (6.11)*, and the summands* $\varphi_{p,j}\left(\alpha_{Y_{l_1,\ldots,l_n}}\right)$ *of* (6.12) *are completely determined by* (6.1)*, or* (6.7)*, for all* $n \in \mathbb{N}$.

Proof. The proof of the formula (6.12) is done by (6.11), (6.1) and (6.7). □

7. FREE PRODUCT C^*-ALGEBRA OF $\left\{M_{\mathcal{P}}^{p,j}\right\}$

In this section, we construct free product C^*-algebra of the system $\{M_{\mathcal{P}}^{p,j}\}_{p\in\mathcal{P},j\in\mathbb{Z}}$, and consider free-distributional data of free reduced words in the C^*-algebra. From this, one can not only study free probability induced by the finite Adele ring, but also apply Adelic analysis as free-probabilistic objects.

In the previous sections, we used concepts and terminology from free probability theory in extended senses (for our commutative structures). In this section, we study (traditional noncommutative) free probability theory on our structures under *free product.*

Let $M_{\mathcal{P}}$ be the finite-Adelic C^*-algebra induced by the finite-Adelic probability space $(\mathcal{M}_{\mathcal{P}},\ \varphi)$ under the representation $(H_{\mathcal{P}},\ \alpha)$, and let

$$\left\{M_{\mathcal{P}}^{p,j} = (M_{\mathcal{P}}, \varphi_{p,j}) : p \in \mathcal{P},\ j \in \mathbb{N}_0\right\} \tag{7.1}$$

be the system of (p,j)-Adelic C^*-probability spaces $M_{\mathcal{P}}^{p,j}$ of (5.2). In this section, we consider *free product C^*-algebra* of the system (7.1).

Recall that, if $\alpha_{q,k}$ are $(+)$-boundary operators $\alpha_{B_k^q}$ in $M_{\mathcal{P}}$, then

$$\varphi_{p,j}\left(\alpha_{q,k}^n\right) = \delta_{j,\{q,p\}}\left(\prod_{r\in\{q,p\}}\left(\frac{1}{r^{k_r}} - \frac{1}{r^{k_r+1}}\right)\right) = \frac{\delta_{j,\{q,p\}}}{n_o n_{p,j}}\phi(n_o), \tag{7.2}$$

for all $n \in \mathbb{N}$, where

$$k_r = \begin{cases} k & \text{if } r = q, \\ j & \text{if } r = p, \end{cases}$$

in $\mathbb{Z}$, and

$$\delta_{j,\{q,p\}} = \begin{cases} \delta_{j,k} & \text{if } q = p, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$n_o = \prod_{r\in\{q,p\}} r^{k_r}, \text{ and } n_{p,j} = \prod_{r\in\{q,p\}} r, \text{ in } \mathbb{N},$$

by (6.7).

Remark that the finite subset $\{q,p\}$ in (7.2) satisfies

$$\{q,p\} = \{q\} \cup \{p\} = \begin{cases} \{q,p\} & \text{if } q \neq p, \\ \{p\} & \text{if } q = p, \end{cases}$$

in $\mathcal{P}$.

Corollary 7.1. *Let $\alpha_{q,k}$ be a $(+)$-boundary operator* (6.9) *in $M_{\mathcal{P}}^{p,j}$, for $p, q \in \mathcal{P}$, $j, k \in \mathbb{N}_0$. Then*

$$\varphi_{p,j}\left(\alpha_{q,k}^{n}\right) = \delta_{j,\{q,p\}}\left(\prod_{r \in \{q,p\}}\left(\frac{1}{r^{k_r}} - \frac{1}{r^{k_r+1}}\right)\right) = \frac{\delta_{j,\{q,p\}}}{n_o n_{p,j}}\phi(n_{p,j}), \tag{7.3}$$

for all $n \in \mathbb{N}$, where $\delta_{j,\{q,p\}}, n_o, n_{p,j}$ are in the sense of (7.2).

Proof. The formula (7.3) is a corollary of (6.7). See (7.2) above. □

By (7.3), one also obtains the following two corollaries.

Corollary 7.2. *Let $M_{\mathcal{P}}^{p,j}$ be a (p,j)-Adelic C^*-probability space, for $p \in \mathcal{P}, j \in \mathbb{N}_0$. Then*

$$\varphi_{p,j}\left(\alpha_{p,j}^{n}\right) = \frac{1}{p^j} - \frac{1}{p^{j+1}} = \frac{1}{p^{j+1}}\phi(p), \tag{7.4}$$

for all $n \in \mathbb{N}$.

Proof. Observe that

$$\varphi_{p,j}\left(\alpha_{p,j}^{n}\right) = \varphi_{p,j}\left(\alpha_{p,j}\right) = \frac{1}{p^j} - \frac{1}{p^{j+1}} = \frac{1}{p^{j+1}}\left(p\left(1 - \frac{1}{p}\right)\right) = \frac{1}{p^{j+1}}\phi(p),$$

by (7.3), for all $n \in \mathbb{N}$. □

Corollary 7.3. *Let $M_{\mathcal{P}}^{p,j}$ be a (p,j)-Adelic C^*-probability space, for $p \in \mathcal{P}, j \in \mathbb{N}_0$. If $p \neq q$ in $\mathcal{P}$, then*

$$\varphi_{p,j}\left(\alpha_{q,k}^{n}\right) = \left(\frac{1}{q^j} - \frac{1}{q^{j+1}}\right)\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right) = \frac{1}{q^{j+1}}\frac{1}{p^{j+1}}\phi(qp), \tag{7.5}$$

for all $n \in \mathbb{N}$.

Proof. Since $p \neq q$ in $\mathcal{P}$, by (7.3), one obtains that

$$\begin{aligned}
\varphi_{p,j}\left(\alpha_{q,k}^{n}\right) &= \varphi_{p,j}\left(\alpha_{q,k}\right) \\
&= \left(\frac{1}{q^k} - \frac{1}{q^{k+1}}\right)\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right) \\
&= \frac{1}{q^k}\frac{1}{p^j}\left(1 - \frac{1}{q}\right)\left(1 - \frac{1}{p}\right) \\
&= \frac{1}{q^{k+1}}\frac{1}{p^{j+1}}\left(q\left(1 - \frac{1}{q}\right)\right)\left(p\left(1 - \frac{1}{p}\right)\right) \\
&= \frac{1}{q^{k+1}p^{j+1}}\left(\phi(q)\right)\left(\phi(p)\right) = \frac{1}{q^{k+1}p^{j+1}}\phi(qp) \\
&= \frac{1}{q^k p^j}\frac{1}{qp}\phi(qp).
\end{aligned}$$

for all $n \in \mathbb{N}$, where

$$n_{\{q,p\}} = q^k p^j, \text{ and } n_{p,j} = qp, \text{ in } \mathbb{N}.$$ □

Indeed, the free-moment formula (7.3) is refined by both (7.4) and (7.5).

7.1. FREE PRODUCT C^*-PROBABILITY SPACES

Let (A_k, φ_k) be arbitrary C^*-probability spaces, consisting of C^*-algebras A_k, and corresponding linear functionals φ_k, for $k \in \Delta$, where Δ is an arbitrary countable (finite or infinite) index set. The *free product C^*-algebra A*,

$$A = \underset{l \in \Delta}{\star} A_l$$

is the C^*-algebra generated by the *noncommutative reduced words* in $\bigcup_{l\in\Delta} A_l$, having a new linear functional

$$\varphi = \underset{l \in \Delta}{\star} \varphi_l.$$

The C^*-algebra A is understood as a *Banach space*,

$$\mathbb{C} \oplus \left(\overset{\infty}{\underset{n=1}{\oplus}} \left(\underset{(i_1,\ldots,i_n)\in alt(\Delta^n)}{\oplus} \left(\overset{n}{\underset{k=1}{\otimes}} A^o_{i_k} \right)\right)\right) \tag{7.6}$$

with

$$A^o_{i_k} = A_{i_k} \ominus \mathbb{C}, \text{ for all } k = 1, \ldots, n,$$

as closed subspaces of A_{i_k}, where

$$alt\,(\Delta^n) = \{(i_1, \ldots, i_n)\,|(i_1, \ldots, i_n) \in \Delta^n, i_1 \neq i_2,\ i_2 \neq i_3, \ldots,\ i_{n-1} \neq i_n\},$$

for all $n \in \mathbb{N}$, and where the *direct product* $\oplus$, and the *tensor product* $\otimes$ are topological on Banach spaces.

In particular, if an element $a \in A$ is a *free* "*reduced*" *word*,

$$\prod_{l=1}^{n} a_{i_l} \text{ in } A,$$

then one can understand a as an equivalent Banach-space vector

$$\eta_a = \overset{n}{\underset{l=1}{\otimes}} a_{i_l} \text{ in the Banach space } A \text{ of (7.6)},$$

contained in a direct summand, $\overset{n}{\underset{k=1}{\otimes}} A^o_{i_k}$ of (7.6). Note that this free reduced word a and its equivalent vector η_a is regarded as an "operator" $\overset{n}{\underset{l=1}{\otimes}} a_{i_l}$ in the C^*-subalgebra

$$\overset{n}{\underset{l=1}{\otimes_{\mathbb{C}}}} A_{i_l} = \mathbb{C} \oplus \left(\overset{n}{\underset{k=1}{\otimes}} A^o_{i_k} \right) \text{ of } A,$$

where $\otimes_{\mathbb{C}}$ means the *tensor product of C^*-algebras*.

We call such a C^*-subalgebra $\overset{n}{\underset{l=1}{\otimes_{\mathbb{C}}}} A_{i_l}$ of A, the *minimal free summand of A containing a*. It is denoted by $A[a]$, i.e., $A[a]$ is the minimal C^*-subalgebra of A containing a as a tensor product operator.

We denote this relation by

$$a \overset{\text{equi}}{=} \underset{l=1}{\overset{n}{\otimes}} a_{i_l} \text{ in } A[a]. \tag{7.7}$$

Notice that the equivalence (7.7) is satisfied in the minimal free summand $A[a]$ of A containing a, "not fully in A, in general".

Remark that, if a is a free reduced word in A, then

$$a^k \overset{\text{equi}}{=} \left(\underset{l=1}{\overset{n}{\otimes}} a_{i_l} \right)^k = \underset{l=1}{\overset{n}{\otimes}} a_{i_l}^k \overset{\text{equi}}{=} \prod_{l=1}^{n} a_{i_l}^k \text{ in } A[a], \tag{7.8}$$

by (7.7) (not in A, in general), for all $k \in \mathbb{N}$.

Let $a = \prod_{l=1}^{n} a_{i_l}$ be a free reduced word in A as above. The power a^k in (7.8) means the k-th power of a in the minimal free summand $A[a]$ of A. To avoid the confusion, we use a different notation $a^{(k)}$, as a new free word (which is either non-reduced or reduced, generally non-reduced),

$$a^{(k)} = \underbrace{a \cdot a \ldots a}_{k\text{-times}}$$

in A.

For example, let $a = a_{i_1} a_{i_2} a_{i_1}$ be a free reduced word for

$$(i_1, i_2, i_1) \in alt\left(\Delta^3\right),$$

as an equivalent vector or a tensor-product operator,

$$a_{i_1} \otimes a_{i_2} \otimes a_{i_1} \text{ in } A[a].$$

Then

$$a^3 \overset{\text{equi}}{=} \left(a_{i_1} \otimes a_{i_2} \otimes a_{i_3}\right)^3 \overset{\text{equi}}{=} a_{i_1}^3 a_{i_2}^3 a_{i_1}^3,$$

in $A[a]$, but

$$\begin{aligned} a^{(3)} &= \left(a_{i_1} a_{i_2} a_{i_1}\right)^{(3)} \\ &= a_{i_1} a_{i_2} a_{i_1} a_{i_1} a_{i_2} a_{i_1} a_{i_1} a_{i_2} a_{i_1} \quad &\text{(non-reduced)} \\ &= a_{i_1} a_{i_2} a_{i_1}^2 a_{i_2} a_{i_1}^2 a_{i_2} a_{i_1}, \quad &\text{(reduced)} \end{aligned}$$

in A, i.e.,

$$a^{(3)} = a_{i_1} a_{i_2} a_{i_1}^2 a_{i_2} a_{i_1}^2 a_{i_2} a_{i_1},$$

is a free reduced word in A.

Similarly, one can use the terms

$$a^* \overset{\text{equi}}{=} \left(\underset{l=1}{\overset{n}{\otimes}} a_{i_l} \right)^* = \underset{l=1}{\overset{n}{\otimes}} a_{i_l}^* \overset{\text{equi}}{=} \prod_{l=1}^{n} a_{i_l}^* \text{ in } A[a],$$

but

$$a^{(*)} = \left(a_{i_1} \ldots a_{i_n}\right)^{(*)} = a_{i_n}^* \ldots a_{i_2}^* a_{i_1}^*, \text{ in } A.$$

However, if a is a free reduced word in A with its length-1, i.e., $a = a_{i_1}$ in A, then

$$a^n = a^{(n)} \text{ in } A[a] = A_{i_1} \subset A,$$

for all $n \in \mathbb{N}$, and

$$a^* = a^{(*)} \text{ in } A[a] = A_{i_1} \subset A.$$

Now, let

$$b = \sum_{l=1}^{n} b_{i_l} \in (A, \varphi). \tag{7.9}$$

We say that an element b of (7.9) is a *free sum in A*, if all summads $b_{i_1}, \ldots, b_{i_n}$ of b are contained in "mutually-distinct" direct summands of a Banach space A of (7.6), as free reduced words (and hence, the summands $b_{i_1}, \ldots, b_{i_n}$ are free from each other in (A, φ)). Then, similar to the above observation, one can realize that

$$b \overset{\text{equi}}{=} \underset{l=1}{\overset{n}{\oplus}} b_{i_l},$$

in the direct summand $\underset{l=1}{\overset{n}{\oplus}} A[b_{i_l}]$ in A, where $A[b_{i_l}]$ are the minimal free summands of A containing b_{i_l}, for all $l = 1, \ldots, n$. We denote $\underset{l=1}{\overset{n}{\oplus}} A[b_{i_l}]$ by $A[b]$, and it is also said to be the *minimal free summand of A containing a free sum b*. Then

$$\begin{aligned} \varphi\left(b^k\right) &\overset{\text{equi}}{=} \varphi\left(\left(\underset{l=1}{\overset{n}{\oplus}} b_{i_l}\right)^k\right) = \varphi\left(\underset{l=1}{\overset{n}{\oplus}} b_{i_l}^k\right) \\ &\overset{\text{equi}}{=} \varphi\left(\sum_{l=1}^{n} b_{i_l}^k\right) = \sum_{l=1}^{n} \varphi\left(b_{i_l}^k\right), \end{aligned} \tag{7.10}$$

on the minimal free summand $A[b]$ of A (not fully on A), for all $k \in \mathbb{N}$.

Here, remark that each summand $\varphi\left(b_{i_l}^k\right)$ of (7.10) satisfies (7.8).

Similar to the free-reduced-word case, if b is a free sum in the sense of (7.9), then one can consider

$$b^{(k)} = \left(\sum_{l=1}^{n} b_{i_l}\right)^{(k)} = \sum_{(l_1, \ldots, l_k) \in \{1, \ldots, n\}^k} \left(b_{i_{l_1}} b_{i_{l_2}} \ldots b_{i_{l_k}}\right),$$

where the summands of $b^{(k)}$ are free words (which are non-reduced in general) in A.

Also, one can distinguish b^* in $A[b]$, and $b^{(*)}$ in A as above.

7.2. THE FINITE-ADELIC C^*-PROBABILITY SPACE $(\mathfrak{M}_{\mathcal{P}},\ \psi)$

Let $M_{\mathcal{P}}^{p,j}$ be (p, j)-Adelic C^*-probability spaces $(M_{\mathcal{P}}, \varphi_{p,j})$, for all $p \in \mathcal{P}, j \in \mathbb{N}_0$, and let

$$\left\{M_{\mathcal{P}}^{p,j} : p \in \mathcal{P},\ j \in \mathbb{N}_0\right\}$$

be the system (7.1) of these C^*-probability spaces.

Construct now the free product C^*-probability space $(\mathfrak{M}_{\mathcal{P}},\ \psi)$ of the system $\{M_{\mathcal{P}}^{p,j}\}_{p,j}$,

$$(\mathfrak{M}_{\mathcal{P}},\ \psi) \overset{def}{=} \underset{p\in\mathcal{P},\ j\in\mathbb{N}_0}{\star} M_{\mathcal{P}}^{p,j} = \left(\underset{p\in\mathcal{P},j\in\mathbb{N}_0}{\star} M_{\mathcal{P}},\ \underset{p\in\mathcal{P},j\in\mathbb{N}_0}{\star} \varphi_{p,j} \right), \tag{7.11}$$

as in Section 7.1 (e.g., see [12] and [14]), where

$$\underset{p\in\mathcal{P},j\in\mathbb{Z}}{\star} M_{\mathcal{P}} = \underbrace{M_{\mathcal{P}} \star\ M_{\mathcal{P}} \star \cdots \star\ M_{\mathcal{P}} \star \dots}_{|\mathcal{P}\times\mathbb{N}_0|\text{-many times}}$$

Definition 7.4. The free product C^*-probability space,

$$\mathfrak{M}_{\mathcal{P}} \overset{denote}{=} (\mathfrak{M}_{\mathcal{P}},\ \psi)$$

of (7.11) is called “the” finite-Adelic (free-product) C^*-probability space.

Remark that, by Section 7.1, even though our finite-Adelic C^*-algebra $M_{\mathcal{P}}$ is commutative, the free product C^*-algebra $\mathfrak{M}_{\mathcal{P}}$ is highly noncommutative, and hence, $(\mathfrak{M}_{\mathcal{P}},\ \psi)$ is a (noncommutative) C^*-probability space (under the traditional sense of free probability theory).

Now, we concentrate on simplest (+)-boundary operators $\alpha_{q,k}$ of $M_{\mathcal{P}}$, in the sense of (6.9) satisfying (7.2), and the corresponding free reduced, or non-reduced words generated by them in the finite-Adelic C^*-probability space $\mathfrak{M}_{\mathcal{P}}$ of (7.11), for all $q \in \mathcal{P}, k \in \mathbb{N}_0$. Again, recall that, by (7.2) (or, by the refined results (7.3) and (7.4), refining (7.2)), one has

$$\varphi_{p,j}\left(\alpha_{p,j}^n\right) = \frac{1}{p^j} - \frac{1}{p^{j+1}} = \frac{1}{p^{j+1}}\phi(p),$$

and

$$\begin{aligned}\varphi_{p,j}\left(\alpha_{q,k}^n\right) &= \left(\frac{1}{q^k} - \frac{1}{q^{k+1}}\right)\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right) \\ &= \frac{1}{q^{k+1}p^{j+1}}\phi(qp),\end{aligned} \tag{7.12}$$

for all $p \neq q \in \mathcal{P}$, and $j, k \in \mathbb{N}_0$, for all $n \in \mathbb{N}$, where ϕ is the Euler totient function.

Let α_{p_l,j_l} be taken from free blocks $M_{\mathcal{P}}^{p_l,j_l}$ of the finite-Adelic C^*-probability space $\mathfrak{M}_{\mathcal{P}}$, for $l = 1, \dots, N$, for $N \in \mathbb{N}$, i.e., choose a subset

$$\left\{\alpha_{p_l,j_l} \in M_{\mathcal{P}}^{p_l,j_l} : l = 1, \dots, N\right\} \text{ in } \mathfrak{M}_{\mathcal{P}}, \tag{7.13}$$

and let

$$T_W^J = \prod_{l=1}^{N} \alpha_{p_l,j_l} \in \mathfrak{M}_{\mathcal{P}} \tag{7.14}$$

induced by the family (7.13), where

$$W = (p_1,\ \dots,\ p_N) \in \mathcal{P}^N,$$

and

$$J = (j_1, \ \ldots, j_N) \in \mathbb{Z}^N.$$

Since $\alpha_{p_l,j_l} = \alpha_{\chi_{B^{p_l}_{j_l}}}$ are in free blocks $M^{p_l,j_l}_{\mathcal{P}}$ in $\mathfrak{M}_{\mathcal{P}}$, for all $l = 1, \ldots, N$, the element T^J_W of (7.14) is understood as a free (reduced, or non-reduced) word,

$$T^J_W = \alpha_{p_1,j_1}\alpha_{p_2,j_2}\ldots\alpha_{p_N,j_N} \text{ in } \mathfrak{M}_{\mathcal{P}}.$$

Assume that either W or J is an alternating N-tuple. Then the free word T^J_W forms a free "reduced" word in $\mathfrak{M}_{\mathcal{P}}$ (See Section 7.1). Thus, if either W or J is an alternating N-tuple, then

$$\left(T^J_W\right)^n \overset{\text{equi}}{=} \left(\underset{l=1}{\overset{N}{\otimes}}\alpha_{p_l,j_l}\right)^n = \underset{l=1}{\overset{N}{\otimes}}\left(\alpha_{p_l,j_l}\right)^n \overset{\text{equi}}{=} \prod_{l=1}^{N}\left(\alpha_{p_l,j_l}\right)^n, \tag{7.15}$$

in the minimal free summand $\mathfrak{M}_{\mathcal{P}}[T^J_W]$ of $\mathfrak{M}_{\mathcal{P}}$ containing T^J_W, for all $n \in \mathbb{N}$.

Remark that

$$\left(T^J_W\right)^{(n)} = \underbrace{T^J_W\ T^J_W \ldots\ T^J_W}_{n\text{-times}} \tag{7.16}$$

is a free (non-reduced) word (in general) in $\mathfrak{M}_{\mathcal{P}}$, for all $n \in \mathbb{N}$. Only if either

$$p_N \neq p_1 \text{ in } \mathcal{P}, \text{ or } j_N \neq j_1 \text{ in } \mathbb{Z},$$

then $\left(T^J_W\right)^{(n)}$ form free reduced words in $\mathfrak{M}_{\mathcal{P}}$, for all $n \in \mathbb{N}$. So, if there is no confusion, one may regard $\left(T^J_W\right)^{(n)}$ as free non-reduced words in $\mathfrak{M}_{\mathcal{P}}$. But, as we discussed in Section 7.1, if T^J_W is a free reduced word with its length-1 in $\mathfrak{M}_{\mathcal{P}}$, then

$$\left(T^J_W\right)^n = \left(T^J_W\right)^{(n)}, \text{ and } \left(T^J_W\right)^* = \left(T^J_W\right)^{(*)},$$

in $\mathfrak{M}_{\mathcal{P}}$.

By (7.15) and (7.16), one obtains the following free-distributional data.

Theorem 7.5. *Let T^J_W be a free random variable* (7.14) *generated by the family* (7.13) *in the finite-Adelic C^*-probability space $\mathfrak{M}_{\mathcal{P}}$, where W and J are in the sense of* (7.14). *If either W or J is an alternating N-tuple, then*

$$\psi\left(\left(T^J_W\right)^n\right) = \prod_{l=1}^{N}\left(\frac{1}{p_l^{j_l}} - \frac{1}{p_l^{j_l+1}}\right) = \prod_{l=1}^{N}\left(\frac{1}{p_l^{j_l+1}}\phi(p_l)\right), \tag{7.17}$$

on its minimal free summand $\mathfrak{M}_{\mathcal{P}}[T^J_W]$ of $\mathfrak{M}_{\mathcal{P}}$, for all $n \in \mathbb{N}$. Moreover, if either $p_N \neq p_1$ in W, or $j_N \neq j_1$ in J, then

$$\psi\left(\left(T^J_W\right)^{(n)}\right) = \prod_{l=1}^{N}\left(\frac{1}{p_l^{j_l}} - \frac{1}{p_l^{j_l+1}}\right)^n = \prod_{l=1}^{N}\left(\frac{1}{p_l^{j_l+1}}\phi(p_l)\right)^n, \tag{7.18}$$

on $\mathfrak{M}_{\mathcal{P}}$, for all $n \in \mathbb{N}$.

Proof. Suppose first that, for a fixed free random variable T_W^J in the sense of (7.14), either W or J is an alternating N-tuple. Then, by (7.13), the operator T_W^J forms a free reduced word in $\mathfrak{M}_{\mathcal{P}}$. Therefore, this free reduced word satisfies the equivalence (7.15), i.e.,

$$\left(T_W^J\right)^n = \prod_{l=1}^{N} (\alpha_{p_l, j_l})^n \text{ for all } n \in \mathbb{N},$$

in the minimal free summand $\mathfrak{M}_{\mathcal{P}}[T_W^J]$ of $\mathfrak{M}_{\mathcal{P}}$. Thus,

$$\begin{aligned}\psi\left(\left(T_W^J\right)^n\right) &= \prod_{l=1}^{N} \varphi_{p_l, j_l}\left(\alpha_{p_l, j_l}^n\right) \\ &= \prod_{l=1}^{N} \varphi_{p_l, j_l}\left(\alpha_{p_l, j_l}\right) = \prod_{l=1}^{N}\left(\frac{1}{p_l^{j_l}} - \frac{1}{p_l^{j_l+1}}\right) = \prod_{l=1}^{N}\left(\frac{1}{p_l^{j_l+1}}\phi(p_l)\right),\end{aligned}$$

on $\mathfrak{M}_{\mathcal{P}}[T_W^J]$, for all $n \in \mathbb{N}$, by (7.12).

Assume now that either W or J is an alternating N-tuple, and assume further that either

$$p_N \neq p_1 \text{ in } \mathcal{P} \text{ or, } j_N \neq j_1 \text{ in } \mathbb{Z}.$$

Then the operator T_W^J forms a free reduced word in $\mathfrak{M}_{\mathcal{P}}$, moreover, $\left(T_W^J\right)^{(n)}$ forms a free “reduced” word in $\mathfrak{M}_{\mathcal{P}}$, too, for all $n \in \mathbb{N}$, by (7.16). Therefore, one obtains that

$$\begin{aligned}\psi\left(\left(T_W^J\right)^{(n)}\right) &= \left(\psi\left(T_W^J\right)\right)^n = \left(\prod_{l=1}^{N} \varphi_{p_l, j_l}\left(\alpha_{p_l, j_l}\right)\right)^n = \prod_{l=1}^{N}\left(\varphi_{p_l, j_l}\left(\alpha_{p_l, j_l}\right)\right)^n \\ &= \prod_{l=1}^{N}\left(\frac{1}{p_l^{j_l}} - \frac{1}{p_l^{j_l+1}}\right)^n = \prod_{l=1}^{N}\left(\frac{1}{p_l^{j_l+1}}\phi(p_l)\right)^n,\end{aligned} \tag{7.19}$$

on $\mathfrak{M}_{\mathcal{P}}$, for all $n \in \mathbb{N}$, by (7.12).

Therefore, the free-distributional data (7.17) and (7.18) hold true. □

Remark that, to satisfy the formula (7.19) in the above proof, the free-“reduced”--word-ness of T_W^J and $\left(T_W^J\right)^{(n)}$ is critical.

Now, let us determine the following family

$$\begin{aligned}\Big\{\alpha_{q_l, k_l} \in M_{\mathcal{P}}^{p_l, j_l} | p_l \neq q_l \text{ in } \mathcal{P}, \text{ and } k_l,\ j_l \in \mathbb{N}_0 \text{ in } \mathbb{Z}, \\ \text{for all } l = 1,\ \ldots,\ N, \text{ for } N \in \mathbb{N}\Big\}\end{aligned} \tag{7.20}$$

of simplest (+)-boundary operators in the free blocks $M_{\mathcal{P}}^{p_l, j_l}$ in the finite-Adelic C^*-probability space $\mathfrak{M}_{\mathcal{P}}$, for $l = 1, \ldots, N$. Simiarly, let

$$U = (q_1,\ \ldots,\ q_N) \in \mathcal{P}^N,$$

and

$$L = (k_1, \ \ldots, \ k_N) \in \mathbb{Z}^N.$$

Here, remark that

$$p_l \neq q_l \text{ in } \mathcal{P}, \text{ for all } l = 1, \ldots, N, \tag{7.21}$$

in (7.20), and furthermore, k_l and j_l are not necessarily identical in $\mathbb{N}_0$, for $l = 1, \ldots, N$. So, the families (7.13) and (7.20) are totally different kinds.

For a newly fixed family (7.20), define a free random variable

$$S_U^L = \prod_{l=1}^{N} \alpha_{q_l, k_l} \in \mathfrak{M}_{\mathcal{P}}, \tag{7.22}$$

where $\alpha_{q_l,k_l} \in M_{\mathcal{P}}^{p_l, j_l}$ are from the family (7.20), for all $l = 1, \ldots, N$.

Theorem 7.6. *Let S_U^L be a free random variable* (7.22) *of the finite-Adelic C^*-probability space $\mathfrak{M}_{\mathcal{P}}$ induced by the family* (7.20)*. Assume that either*

$$W = (p_1, \ \ldots, \ p_N) \in \mathcal{P}^N,$$

or

$$J = (j_1, \ \ldots, \ j_N) \in \mathbb{Z}^N$$

is an alternating N-tuple. Then

$$\psi\left(\left(S_U^L\right)^n\right) = \prod_{l=1}^{N} \left(\frac{1}{q_l^{k_l}} - \frac{1}{q_l^{k_l+1}}\right)\left(\frac{1}{p_l^{j_l}} - \frac{1}{p_l^{j_l+1}}\right) = \prod_{l=1}^{N} \left(\frac{\phi(q_l p_l)}{q_l^{k_l+1} p_l^{j_l+1}}\right), \tag{7.23}$$

for all $n \in \mathbb{N}$.

Moreover, if either $p_N \neq p_1$ in $\mathcal{P}$, and W is alternating, or if $j_N \neq j_1$ in $\mathbb{Z}$, and J is alternating, then

$$\psi\left(\left(S_U^L\right)^{(n)}\right) = \left(\prod_{l=1}^{N} \left(\frac{1}{q_l^{k_l}} - \frac{1}{q_l^{k_l+1}}\right)\left(\frac{1}{p_l^{j_l}} - \frac{1}{p_l^{j_l+1}}\right)\right)^n = \left(\prod_{l=1}^{N} \frac{\phi(q_l p_l)}{q_l^{k_l+1} p_l^{j_l+1}}\right)^n \tag{7.24}$$

for all $n \in \mathbb{N}$.

Proof. First, assume that S_U^L is in the sense of (7.22) generated by a family (7.20) in $\mathfrak{M}_{\mathcal{P}}$, where either W or J is alternating. Then the operator S_U^L forms a free reduced word in $\mathfrak{M}_{\mathcal{P}}$, satisfying that

$$\left(S_U^L\right)^n \overset{\text{equi}}{=} \left(\underset{l=1}{\overset{N}{\otimes}} \alpha_{q_l, k_l}\right)^n = \left(\underset{l=1}{\overset{N}{\otimes}} \left(\alpha_{q_l, k_l}\right)^n\right) \overset{\text{equi}}{=} \prod_{l=1}^{N} \alpha_{q_l, k_l}^n \tag{7.25}$$

in its minimal free summand $\mathfrak{M}_{\mathcal{P}}\left[S_U^L\right]$ of $\mathfrak{M}_{\mathcal{P}}$, for all $n \in \mathbb{N}$. Thus, one can get that

$$\psi\left(\left(S_U^L\right)^n\right) = \psi\left(\prod_{l=1}^{N} \alpha_{q_l,k_l}^n\right) = \prod_{l=1}^{N} \varphi_{p_l,j_l}\left(\alpha_{q_l,k_l}^n\right)$$

by (7.20), (7.22) and (7.25)

$$= \prod_{l=1}^{N} \varphi_{p_l,j_l}\left(\alpha_{q_l,k_l}\right)$$

$$= \prod_{l=1}^{N} \left(\frac{1}{q_l^{k_l}} - \frac{1}{q_l^{k_l+1}}\right)\left(\frac{1}{p_l^{j_l}} - \frac{1}{p_l^{j_l+1}}\right)$$

by (7.21) and (7.12)

$$= \prod_{l=1}^{N} \left(\frac{\phi(q_l p_l)}{q_l^{k_l+1} p_l^{j_l+1}}\right),$$

by (7.12). Thus, we obtain the free-distributional data (7.23) of S_U^L in $\mathfrak{M}_{\mathcal{P}}\left[S_U^L\right]$.

Now, assume either W or J is an alternating N-tuple. Also, suppose either $p_N \neq p_1$ in $\mathcal{P}$, or $j_N \neq j_1$ in $\mathbb{Z}$. Then the operator S_U^L is a free reduced word in $\mathfrak{M}_{\mathcal{P}}$, moreover, $\left(S_U^L\right)^{(n)}$ form free reduced words in $\mathfrak{M}_{\mathcal{P}}$, for all $n \in \mathbb{N}$. Thus,

$$\begin{aligned} \psi\left(\left(S_U^L\right)^{(n)}\right) &= \left(\psi\left(S_U^L\right)\right)^n \\ &= \left(\prod_{l=1}^{N} \varphi_{p_l,j_l}\left(\alpha_{q_l,k_l}\right)\right)^n \\ &= \left(\prod_{l=1}^{N} \left(\frac{1}{q_l^{k_l}} - \frac{1}{q_l^{k_l+1}}\right)\left(\frac{1}{p_l^{j_l}} - \frac{1}{p_l^{j_l+1}}\right)\right)^n \\ &= \left(\prod_{l=1}^{N} \frac{\phi(q_l p_l)}{q_l^{k_l+1} p_l^{j_l+1}}\right)^n, \end{aligned} \tag{7.26}$$

for all $n \in \mathbb{N}$. Therefore, we obtain the free-distributional data (7.24) by (7.26). □

Now, let us fix the family (7.13), and construct a free random variable $T_{W,J}$ in $\mathfrak{M}_{\mathcal{P}}$ by

$$T_{W,J} = \sum_{l=1}^{N} \alpha_{p_l,j_l}. \tag{7.27}$$

Theorem 7.7. *Let $T_{W,J}$ be a free random variable* (7.27) *generated by the family* (7.13) *in the finite-Adelic C^*-probability space $\mathfrak{M}_{\mathcal{P}}$. Suppose either*

$$W = (p_1, \ \ldots, \ p_N), \ \textit{or}\ J = (j_1, \ \ldots, \ j_N)$$

is an N-tuple of mutually-distinct entries in $\mathcal{P}$, respectively, in $\mathbb{Z}$. Then

$$\psi\left((T_{W,J})^n\right) = \sum_{l=1}^{N} \left(\frac{1}{p_l^{j_l}} - \frac{1}{p_l^{j_l+1}} \right) = \sum_{l=1}^{N} \frac{\phi(p_l)}{p_l^{j_l+1}}, \tag{7.28}$$

on the minimal free summand $\mathfrak{M}_{\mathcal{P}}[T_{W,J}]$ of $\mathfrak{M}_{\mathcal{P}}$ containing $T_{W,J}$, for all $n \in \mathbb{N}$.

Proof. Suppose $T_{W,J}$ be in the sense of (7.27) in $\mathfrak{M}_{\mathcal{P}}$, and assume that either W or J is an N-tuple consisting of mutually-distinct entries. Then the operator $T_{W,J}$ forms a free sum in $\mathfrak{M}_{\mathcal{P}}$, satisfying

$$\begin{aligned} (T_{W,J})^n &\overset{\text{equi}}{=} \left(\underset{l=1}{\overset{N}{\oplus}} \alpha_{p_l,j_l} \right)^n = \underset{l=1}{\overset{N}{\oplus}} \alpha_{p_l,j_l}^n \\ &\overset{\text{equi}}{=} \sum_{l=1}^{N} \alpha_{p_l,j_l}^n = \sum_{l=1}^{N} \alpha_{p_l,j_l} = T_{W,J}, \end{aligned} \tag{7.29}$$

in $\mathfrak{M}_{\mathcal{P}}$, for all $n \in \mathbb{N}$.

Thus, one can have that

$$\begin{aligned} \psi\left((T_{W,J})^n\right) = \psi\left(T_{W,J}\right) &= \psi\left(\sum_{l=1}^{N} \alpha_{p_l,j_l} \right) \\ &= \sum_{l=1}^{N} \psi\left(\alpha_{p_l,j_l}\right) = \sum_{l=1}^{N} \varphi_{p_l,j_l}\left(\alpha_{p_l,j_l}\right) \end{aligned}$$

by (7.29)

$$= \sum_{l=1}^{N} \left(\frac{1}{p_l^{j_l}} - \frac{1}{p_l^{j_l+1}} \right) = \sum_{l=1}^{N} \frac{\phi(p_l)}{p_l^{j_l+1}},$$

for all $n \in \mathbb{N}$. Therefore, we obtain the free-moment formula (7.28) for $T_{W,J}$ in $\mathfrak{M}_{\mathcal{P}}\left[T_{W,J}\right]$. □

Now, let us fix a family (7.20), and construct a free random variable $S_{U,L}$,

$$S_{U,L} = \sum_{l=1}^{N} \alpha_{q_l,k_l} \in \mathfrak{M}_{\mathcal{P}}, \tag{7.30}$$

where

$$\alpha_{q_l,k_l} \in M_{\mathcal{P}}^{p_l,j_l} \text{ in } \mathfrak{M}_{\mathcal{P}}, \text{ for all } l = 1, \ldots, N,$$

where

$$U = (q_1, \ldots, q_N) \in \mathcal{P}^N, \quad \text{and} \quad L = (k_1, \ \ldots, \ k_N) \in \mathbb{Z}^N,$$

given in (7.20).

Theorem 7.8. *Let $S_{U,L}$ be a free random variable* (7.30) *in the finite-Adelic C^*-probability space $\mathfrak{M}_{\mathcal{P}}$. Assume that either*

$$W = (p_1, \ \ldots, p_N) \in \mathcal{P}^N, \ \textit{or} \ J = (j_1, \ \ldots, j_N) \in \mathbb{Z}^N$$

is an N-tuple of mutually-distinct entries. Then

$$\psi\left((S_{U,L})^n\right) = \sum_{l=1}^{N} \left(\frac{1}{q_l^{k_l}} - \frac{1}{q_l^{k_l+1}}\right)\left(\frac{1}{p_l^{j_l}} - \frac{1}{p_l^{j_l+1}}\right) = \sum_{l=1}^{N} \frac{\phi(q_l p_l)}{q_l^{k_l+1} p_l^{j_l+1}}, \tag{7.31}$$

on the minimal free summand $\mathfrak{M}_{\mathcal{P}}[S_{U,L}]$ of $\mathfrak{M}_{\mathcal{P}}$ containing $S_{U,L}$, for all $n \in \mathbb{N}$.

Proof. By the assumption that either W or J is an N-tuple consisting of mutually distinct entries, the operator $S_{U,L}$ forms a free sum in $\mathfrak{M}_{\mathcal{P}}$ satisfying that

$$\begin{aligned} (S_{U,L})^n &\overset{\text{equi}}{=} \left(\underset{l=1}{\overset{N}{\oplus}} \alpha_{q_l,k_l}\right)^n = \underset{l=1}{\overset{N}{\oplus}} \alpha_{q_l,k_l}^n \\ &\overset{\text{equi}}{=} \sum_{l=1}^{N} \alpha_{q_l,k_l}^n = \sum_{l=1}^{N} \alpha_{q_l,k_l} = S_{U,L}, \end{aligned} \tag{7.32}$$

in its minimal free summand $\mathfrak{M}_{\mathcal{P}}\left[S_{U,L}\right]$ of $\mathfrak{M}_{\mathcal{P}}$, for all $n \in \mathbb{N}$. And hence, one obtains that

$$\begin{aligned} \psi\left((S_{U,L})^n\right) &= \psi\left(S_{U,L}\right) \\ &= \sum_{l=1}^{N} \psi\left(\alpha_{q_l,k_l}\right) = \sum_{l=1}^{N} \varphi_{p_l,j_l}\left(\alpha_{q_l,k_l}\right) \end{aligned}$$

by (7.20)

$$= \sum_{l=1}^{N} \left(\frac{1}{q_l^{k_l}} - \frac{1}{q_l^{k_l+1}}\right)\left(\frac{1}{p_l^{j_l}} - \frac{1}{p_l^{j_l+1}}\right) = \sum_{l=1}^{N} \frac{\phi(q_l p_l)}{q_l^{k_l+1} p_l^{j_l+1}},$$

by (7.12), for all $n \in \mathbb{N}$. Therefore, we obtain the free-distributional data (7.31) for $S_{U,L}$ in $\mathfrak{M}_{\mathcal{P}}[S_{U,L}]$. □

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Ilwoo Cho
choilwoo@sau.edu

Saint Ambrose University
Department of Mathematics and Statistics
421 Ambrose Hall, 518 W. Locust St.
Davenport, Iowa, 52803, USA

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