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2-SPLITTABLE AND CORDIAL GRAPHS

Abstract. E. Miller and G.E. Stevens proved in [5] the existence of certain families of 2-splittable caterpillars. In this paper we characterize other families of 2-splittable caterpillars. Moreover, we show that for some of them there exists a friendly labeling inducing two isomorphic subgraphs.

Keywords: cordial graphs, 2-splittable graphs.

Mathematics Subject Classification: 05C78.

1. INTRODUCTION

All graphs considered in this paper are simple, finite and undirected. We use the standard terminology and notation of graph theory. Denote by $|G|$ the order of graph G, $|V(G)|$, and by $\|G\|$ the size of G, $|E(G)|$.

A graph G is *2-splittable* if its edge set can be partitioned into two subsets so that the induced subgraphs are isomorphic. This is equivalent to coloring the edges of the graph red and green so that the red subgraph is isomorphic to the green one ([5]). However we can color the graph by 0 and 1.

The problem of deciding whether a given graph is 2-splittable has been considered in several papers (see for example [5], [7]). This problem, even if restricted to trees, is NP-complete [4].

For a graph $G = (V, E)$ define a binary labelling $c : V(G) \to \mathbb{Z}_2$. Under such a labelling c, we let $v_c(i)$ denote the number of vertices in G that are labelled i. The labelling c is said to be *friendly* if $|v_c(1) - v_c(0)| \leq 1$. The labelling $c : V(G) \to \mathbb{Z}_2$ induces an edge labelling $c^* : E(G) \to \mathbb{Z}_2$ defined by $c^*(xy) = |c(x) + c(y)|$ ([6]). Let $E_{c^*}(j) = \{\{x, y\} : \{x, y\} \in E \text{ and } c(x) + c(y) = j\}$ and $e_{c^*}(j) = |E_{c^*}(j)|$. A graph is said to be *cordial* if it admits such a friendly labelling c that $|e_{c^*}(1) - e_{c^*}(0)| \leq 1$. Cordial labelling was introduced by I. Cahit [1] as a weakened version of graceful labelling and harmonious labelling.

He also proved the following theorem:

Theorem 1.1 ([1]). *The following families of graphs are cordial:*

1. *trees,*
2. *complete graphs K_n if and only if $n \leq 3$,*
3. *complete bipartite graphs $K_{n,m}$ for all m and n,*
4. *cycles C_n if and only if $n \not\equiv 2 \bmod 4$.*

However, N. Cairnie and K. Edwards proved that in a general case the problem of deciding whether or not a graph G is cordial is NP-complete [2].

We say that a labelling c induces two isomorphic subgraphs if the subgraph with edges of label 1 is isomorphic to the subgraph with edges with label 0.

We are investigating the question of whether a 2-splittable graph can have a friendly labelling, which induces two isomorphic subgraphs. Obviously a necessary condition for the existence of such a friendly labelling of a graph G is that G is cordial. It is worth mentioning that there exists 2-splittable graphs (Fig. 1) for which there does not exist a friendly labelling inducing two isomorphic graphs, while they are cordial (by Theorem 1.1). Moreover, there exists some 2-splittable graphs which are not even cordial (for instance a cycle C_n for $n \equiv 2 \bmod 4$).

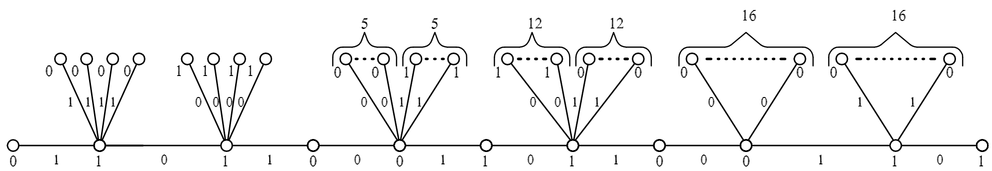

Fig. 1. Example of a 2-splittable catrepillar

As in [5] a *caterpillar* is a tree in which all pedant edges are incident with vertices along a single path called the *spine*. The consecutive vertices of spine we denote as $x_1, x_2, \ldots, x_k$. Pedant edges adjacent with the spine are *legs*. We assume that the first (x_1) and the last (x_k) vertices of the spine has no legs and we will call these vertices the *head* and *tail*. We will refer to the edges of the spine incident with these vertices as the *head* and *tail edges*. We call the interior vertices incident with legs of the spine *joints* and if a vertex has no legs incident with it we call it a *nub*. For a joint x_i the end vertices of its d_i legs are denoted by $v_1^i, v_2^i, \ldots, v_{d_i}^i$ (Fig. 2). A caterpillar T is an *r-legged caterpillar* if every joint has degree $r+2$ (i.e. every joint is incident with exactly r legs).

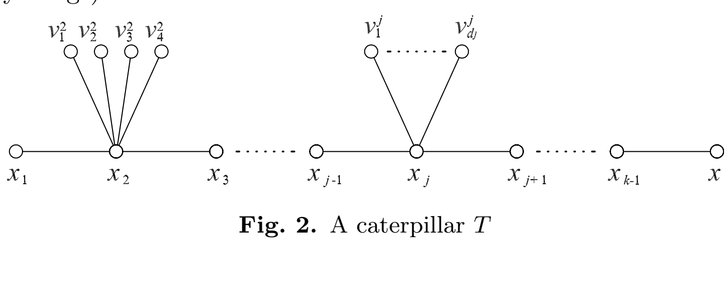

Fig. 2. A caterpillar T

A graph S is *a sun* if the set of vertices of degree at least two induces a cycle $C_n = x_1, x_2, \dots, x_n$. Pendant edges adjacent with the cycle are called *rays*. As was done above we call the interior vertices incident with rays of the cycle *joints* and if a vertex has no rays incident with it we call it a *nub*. For a joint x_i the end vertices of its d_i rays are denoted by $v_1^i, v_2^i, \dots, v_{d_i}^i$ (Fig. 3).

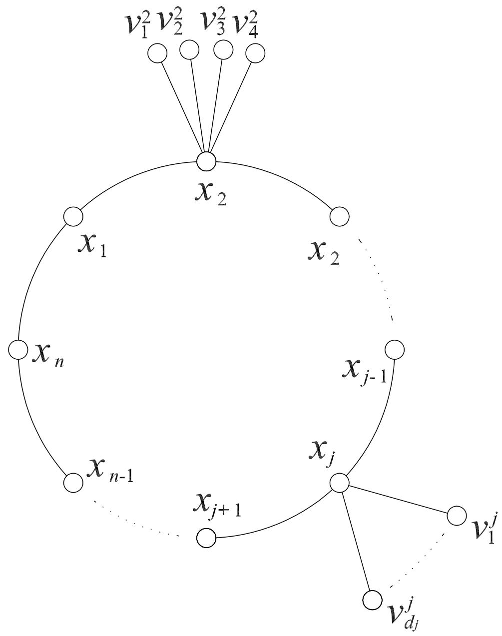

Fig. 3. A sun S

It is obvious that in order to be 2-splittable a graph has to have an even number of edges. E. Miller and G.E. Stevens have proved in [5] that there are some families of caterpillars for which this condition is also sufficient.

Theorem 1.2. *All even sized single legged caterpillars and double legged caterpillars are 2-splittable.*

In this paper we show some new families of 2-splittable caterpillars and prove that for some of them there exists a friendly labelling inducing two isomorphic graphs.

2. MAIN RESULTS

Not all caterpillars with an even size are 2-splittable. For instance in Figure 4, we present an even sized caterpillar with two joints with degrees 6 and 9 which is not 2-splittable. Note that the distance between the joints of degree 6 and 9 is one. It seems that the distance between joints is also important in the general case.

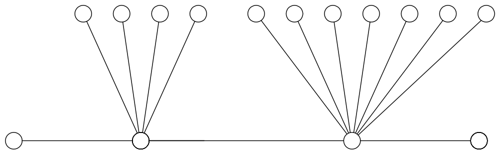

Fig. 4. A non 2-splittable caterpillar

Theorem 2.1. *A caterpillar T of an even size with a spine $x_1, x_2, \dots, x_k$ is 2-splittable with the head and tail edges belonging to different label classes, if the following conditions are satisfied:*

1. *if consecutive joints have different degrees then the distance between them is at least 2, and*
2. *for each odd number d so that there is a joint with the degree d, there exists an even number of joints with the degree d.*

Moreover, if there exist only joints with an even degree, and there are no joints at mutual distance one, then we can define a friendly labelling $f : V(T) \to \mathbb{Z}_2$ inducing an edge labelling $f^* : E(T) \to \mathbb{Z}_2$ defined by $f^*(xy) = |f(x) + f(y)|$. Furthermore, if $k \equiv 1 \bmod 4$, then $f(x_1) = f(x_k)$.

Proof. Notice that because we have an even number of joints with an odd degree, then there is an even number of edges in the spine $x_1, x_2, \dots, x_k$, and it follows that k is odd. We can split the graph by alternately labelling the edges along the spine 0 and 1. Let $f(x_1) = 0$, $f(x_2) = 1$, $f(x_i) = f(x_{i-2}) + 1$ for $i = 3, \dots, k$. It follows that $f^*(x_i x_{i+1}) = f^*(x_{i+1} x_{i+2}) + 1$ for $i = 1, \dots, k-2$. Moreover $f^*(x_1 x_2) = 1$, $f^*(x_{k-1} x_k) = 0$ and if $k \equiv 1 \bmod 4$ then $f(x_1) = f(x_k)$. Let $h_i = \lceil \frac{d_i}{2} \rceil$ for all x_i. We start labelling end vertices of legs at the first joint and we use the following procedure:

Let x_i be a joint. If $d_{i+1} = 0$, then let $f(v_1^i) = \dots = f(v_{h_i-1}^i) = f(x_{i-1})$, $f(v_{h_i+1}^i) = \dots = f(v_{d^i}^i) = f(x_{i+1})$. If now d_i is even or among vertices $x_1, x_2, \dots, x_{i-1}$ there exists an even number of joints such that they have the same degree as x_i, then let $f(v_{h_i}^i) = f(x_{i-1})$. Assume now that d_i is odd and in vertices $x_1, x_2, \dots, x_{i-1}$ there exists an odd number of joints such that they have the same degree d_i. Let x_j be such that $d_j = d_i$ and there does not exist x_r such that $d_r = d_i$ for $r = j+1, \dots, i-1$. Now let $f(v_{h_i}^i)$ have a label such that $f^*(v_{h_i}^i x_i) = f^*(v_{h_j}^j x_j) + 1$.

Assume now $d_{i+1} = d_i$ then let $f(v_1^i) = \dots = f(v_{d_i}^i) = f(x_{i-1})$ and $f(v_1^{i+1}) = \dots = f(v_{d_{i+1}}^{i+1}) = f(x_i)$. Obviously the labelling $f^* : E(T) \to \mathbb{Z}_2$ induces two isomorphic subgraphs and one of them has label 1, whereas the other one has label 0 (see Fig. 5).

Notice that the spine of the caterpillar T is friendly labeled. Moreover, if there exists only joints with even degrees, and there does not exist joints at mutual distance one, then for any joint x_i we have $h_i = \frac{d_i}{2}$ and $f(v_1^i) = \dots = f(v_{h_i}^i) = f(x_{i-1})$, $f(v_{h_i+1}^i) = \dots = f(v_{d^i}^i) = f(x_{i+1}) = f(x_{i-1}) + 1$. It implies that f is a friendly labelling. $\square$

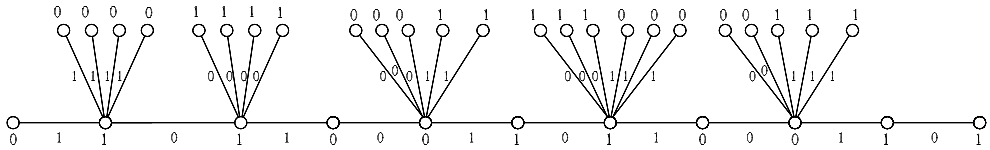

Fig. 5. A labelling of a 2-splittable caterpillar

Notice that there exist caterpillars with consecutive joints at mutual distance one having the same degree such that there does not exist friendly labelling inducing two isomorphic subgraphs, see Figure 1.

It is worth noticing that there are infinitely many 2-splittable caterpillars which have consecutive joints with different degrees with distance one between them (see Fig. 6).

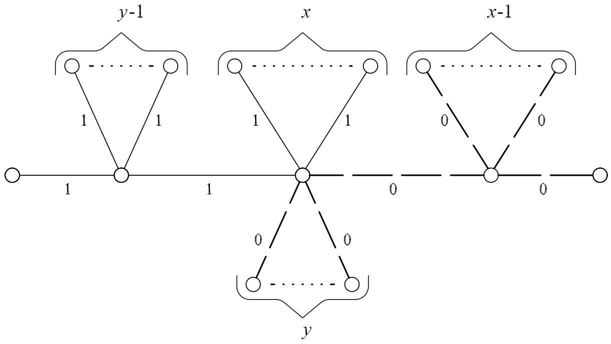

Fig. 6. A 2-splittable caterpillar

Remember that by Theorem 1.1 for a cycle C_n with $n \equiv 2 \bmod 4$ there does not exist a friendly labelling inducing two isomorphic subgraphs.

Theorem 2.1 implies easily:

Theorem 2.2. *An even sized sun S with a cycle $C_n = x_1, x_2, \ldots, x_n$ is 2-splittable, if the following conditions are satisfied:*

1. *if consecutive joints have different degrees then the distance between them is at least 2, and*
2. *for each odd number d so that there is a joint with the degree d, there exists an even number of joints with the degree d.*

Moreover, if there are only joints with even degrees, any two of them at distance at least two apart and $n \equiv 0 \bmod 4$, then we can define a friendly labelling $f : V(S) \to \mathbb{Z}_2$ inducing an edge labelling $f^* : E(S) \to \mathbb{Z}_2$ defined by $f^*(xy) = |f(x) + f(y)|$.

Proof. Notice that if there exists any nub v in the sun S then we can split that sun at the nub v and we obtain a caterpillar T satisfying the assumptions of Theorem 2.1.

It implies that the caterpillar T is 2-splittable with the head and tail edges belonging to different label classes. Since we do not care now about labels of vertices, we can glue together the head and tail of T and see that the sun S is 2-splittable (see Fig. 5 and 7). Moreover if we use the labelling (of vertices) defined in Theorem 2.1, then if $n \equiv 0 \bmod 4$ and there exist only joints with even degrees, and any two of them are at distance at least two apart, then f is a friendly labelling of S.

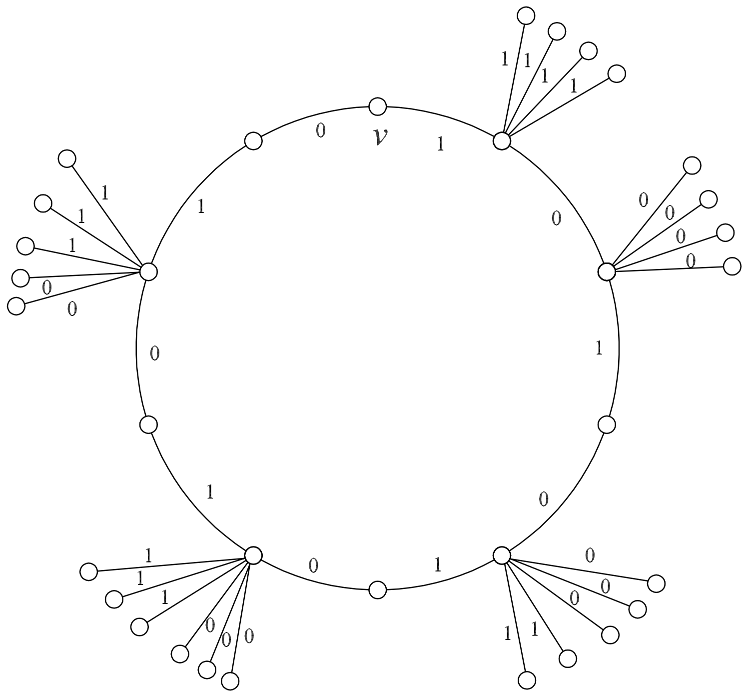

Fig. 7. A 2-splittable sun

Therefore, we can assume that there is no nub in the sun S. Since the size of S is even it follows by our assumption that there is an even number of joints and all of them have the same degree. Define now $f^* : E(S) \to \mathbb{Z}_2$ as follows: $f^*(x_1x_2) = 0$, $f^*(x_{i+1}x_{i+2}) = f^*(x_ix_{i+1}) + 1$, $f^*(v_1^ix_i) = \ldots = f^*(v_{d_i}^ix_i) = f^*(x_ix_{i+1})$ for all i. We can see that the edge labelling f^* induces two isomorphic subgraphs S_0, S_1 of S of labels 0 and 1, respectively, and the theorem is proved. $\square$

3. CONCLUSIONS

It is obvious that the class of cordial graphs is not equivalent to the class of 2-spittable graphs. It seems to be interesting to characterize all families of 2-splittable graphs such that there exists a friendly coloring inducing two isomorphic subgraphs.

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