

**A NOTE  
ON NONLOCAL DISCRETE PROBLEMS  
INVOLVING  
SIGN-CHANGING KIRCHHOFF FUNCTIONS**

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**Abstract.** In this note, we establish a multiplicity theorem for a nonlocal discrete problem of the type

$$\begin{cases} -\left(a \sum_{m=1}^{n+1} |x_m - x_{m-1}|^2 + b\right) (x_{k+1} - 2x_k + x_{k-1}) = h_k(x_k), & k = 1, \dots, n, \\ x_0 = x_{n+1} = 0 \end{cases}$$

assuming  $a > 0$  and (for the first time)  $b < 0$ .

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## 1. INTRODUCTION

If  $\Omega$  is a bounded domain of  $\mathbb{R}^n$  and  $K : [0, +\infty[ \rightarrow \mathbb{R}$ ,  $\varphi : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  are two given functions, the nonlocal problem

$$\begin{cases} -K\left(\int_{\Omega} |\nabla u(x)|^2\right) \Delta u = \varphi(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is certainly among the most studied ones in today's nonlinear analysis (we refer to [4] for an introduction to the subject).

In checking the relevant literature, one can realize that, in the majority of the papers, one assumes  $K(t) = at + b$  with  $a > 0$  and  $b \geq 0$  and, in any case, that the Kirchhoff function  $K$  is assumed to be, in particular, continuous and non-negative in  $[0, +\infty[$ .

However, it is natural to ask what happens when at least one of these properties fails.

The case where  $K$  can be discontinuous in  $[0, +\infty[$  has been considered for the first time in [5], for  $n = 1$ , and then in [6] for the general case (see also [1–3]). In these papers, however,  $K$  is non-negative.

The papers dealing with a sign-changing function  $K$  are more numerous, but in each of them it is assumed that  $K(t) = at + b$  with  $a < 0$  and  $b \geq 0$ . The first of these papers was [8].

In the present very short note, we are interested in the discrete counterpart of the above problem. That is to say, given  $n$  continuous functions  $f_k : \mathbb{R} \rightarrow \mathbb{R}$  ( $k = 1, \dots, n$ ), we deal with the problem

$$\begin{cases} -K \left( \sum_{h=1}^{n+1} |x_h - x_{h-1}|^2 \right) (x_{k+1} - 2x_k + x_{k-1}) = f_k(x_k), & k = 1, \dots, n, \\ x_0 = x_{n+1} = 0. \end{cases}$$

Also for this discrete problem, we can repeat what we said before, even in a stronger way: it seems that in each paper on the subject, the function  $K$  is continuous and non-negative in  $[0, +\infty[$ .

Our aim is to establish a multiplicity result for this problem where (for the first time) the Kirchhoff function  $K$  changes sign.

## 2. RESULTS

Before stating our result, we recall the following two theorems which will be key tools used in our proof.

**Theorem 2.1** ([7]). *Let  $X$  be a topological space, let  $Y$  be a convex set in a topological vector space and let  $h : X \times Y \rightarrow \mathbb{R}$  be lower semicontinuous and inf-compact in  $X$ , and continuous and quasi-concave in  $Y$ . Also, assume that*

$$\sup_Y \inf_X h < \inf_X \sup_Y h.$$

Moreover, let  $\varphi : X \rightarrow \mathbb{R}$  be a lower semicontinuous function such that

$$\sup_X \varphi - \inf_X \varphi < \inf_X \sup_Y h - \sup_Y \inf_X h.$$

Then, for each convex set  $S \subseteq Y$ , dense in  $Y$ , there exists  $\tilde{y} \in S$  such that the function  $h(\cdot, \tilde{y}) + \varphi(\cdot)$  has at least two global minima.

**Theorem 2.2** ([7]). *Let  $X$  be a topological space, let  $H$  be a real Hilbert, let  $Y$  be a closed ball in  $H$  centered at  $0$ , and let  $Q : X \rightarrow \mathbb{R}$ ,  $\psi : X \rightarrow H$ . Assume that the functional  $x \mapsto Q(x) - \langle \psi(x), y \rangle$  is lower semicontinuous for each  $y \in Y$ , while the functional  $x \mapsto Q(x) - \langle \psi(x), y_0 \rangle$  is inf-compact for some  $y_0 \in Y$ . Moreover, assume that, for each  $x \in X$ , there exists  $u \in X$  such that*

$$Q(x) = Q(u)$$

and

$$\psi(x) = -\psi(u).$$

Finally, assume that there is no global minimum of  $Q$  at which  $\psi$  vanishes. Then, we have

$$\sup_{y \in Y} \inf_{x \in X} (Q(x) - \langle \psi(x), y \rangle) < \inf_{x \in X} \sup_{y \in Y} (Q(x) - \langle \psi(x), y \rangle).$$

Our main result is as follows:

**Theorem 2.3.** *Let  $K : [0, +\infty[ \rightarrow \mathbb{R}$ ,  $f_1, \dots, f_n : \mathbb{R} \rightarrow \mathbb{R}$  be  $n + 1$  continuous functions satisfying the following conditions:*

- (a)  $\inf_{t>0} \int_0^t K(s)ds < 0$  and  $\liminf_{t \rightarrow +\infty} \frac{\int_0^t K(s)ds}{t} > 0$ ,
- (b)  $\limsup_{|t| \rightarrow +\infty} \frac{|\int_0^t f_k(s)ds|}{t^2} < +\infty$  for each  $k = 1, \dots, n$ ,
- (c) for each  $k = 1, \dots, n$ , the function  $t \rightarrow \int_0^t f_k(s)ds$  is odd and vanishes only at 0.

Then, for each  $r > 0$ , there exists a number  $\delta > 0$  with the following property: for every  $n$ -tuple of continuous functions  $g_1, \dots, g_n : \mathbb{R} \rightarrow \mathbb{R}$  satisfying

$$\max_{1 \leq k \leq n} \left( \sup_{t \in \mathbb{R}} \int_0^t g_k(s)ds - \inf_{t \in \mathbb{R}} \int_0^t g_k(s)ds \right) < \delta,$$

there exists  $(\tilde{\mu}_1, \dots, \tilde{\mu}_n) \in \mathbb{R}^n$ , with  $\sum_{k=1}^n |\tilde{\mu}_k|^2 < r^2$ , such that the problem

$$\begin{cases} -K(\sum_{h=1}^{n+1} |x_h - x_{h-1}|^2)(x_{k+1} - 2x_k + x_{k-1}) = g_k(x_k) + \tilde{\mu}_k f_k(x_k), & k=1, \dots, n, \\ x_0 = x_{n+1} = 0 \end{cases}$$

has at least three solutions.

*Proof.* Fix  $r > 0$ . First, we are going to apply Theorem 2.2. In this connection, take

$$X = \{(x_0, x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+2} : x_0 = x_{n+1} = 0\},$$

with the scalar product

$$\langle x, y \rangle_1 = \sum_{k=1}^{n+1} (x_k - x_{k-1})(y_k - y_{k-1}).$$

We denote by  $\langle \cdot, \cdot \rangle_2$  the usual scalar product on  $\mathbb{R}^n$ , that is,

$$\langle x, y \rangle_2 = \sum_{k=1}^n x_k y_k.$$

Fix  $\gamma > 0$  so that

$$\|x\|_2 \leq \gamma \|x\|_1 \tag{2.1}$$

for all  $x \in X$ . Consider the functions  $Q : X \rightarrow \mathbb{R}$  and  $\psi : X \rightarrow \mathbb{R}^n$  defined by

$$Q(x) = \frac{1}{2} \int_0^{\|x\|_1^2} K(s)ds$$

and

$$\psi(x) = \left( \int_0^{x_1} f_1(s)ds, \dots, \int_0^{x_n} f_n(s)ds \right)$$

for all  $x \in X$ . Fix  $\mu \in \mathbb{R}^n$ . In view of (a) and (b), there exist  $\eta_1, \eta_2, \eta_3 > 0$  such that

$$\int_0^t K(s)ds \geq \eta_1 t - \eta_2 \tag{2.2}$$

for all  $t \geq 0$  and

$$\left| \int_0^t f_k(s)ds \right| \leq \eta_3(t^2 + 1) \tag{2.3}$$

for all  $t \in \mathbb{R}, k = 1, \dots, n$ . Fix  $x \in X$ . Using (2.2) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} Q(x) - \langle \psi(x), \mu \rangle_2 &\geq \frac{1}{2}\eta_1 \|x\|_1^2 - \frac{1}{2}\eta_2 - |\langle \psi(x), \mu \rangle_2| \\ &\geq \frac{1}{2}\eta_1 \|x\|_1^2 - \frac{1}{2}\eta_2 - \|\mu\|_2 \|\psi(x)\|_2. \end{aligned} \tag{2.4}$$

On the other hand, in view of (2.3), for each  $k = 1, \dots, n$ , we have

$$\left| \int_0^{x_k} f_k(s)ds \right| \leq \eta_3(|x_k|^2 + 1)$$

and hence

$$\|\psi(x)\|_2 \leq \eta_3 \sqrt{\sum_{k=1}^n (|x_k|^2 + 1)^2} \leq \eta_3 \left( \sum_{k=1}^n |x_k|^2 + n \right). \tag{2.5}$$

Putting (2.1), (2.4) and (2.5) together, we get

$$Q(x) - \langle \psi(x), \mu \rangle_2 \geq \frac{1}{2}\eta_1 \|x\|_1^2 - \|\mu\|_2 \eta_3 (\gamma^2 \|x\|_1^2 + n) - \frac{1}{2}\eta_2. \tag{2.6}$$

Now, fix  $\sigma > 0$  so that

$$\sigma < \min \left\{ \frac{\eta_1}{2\eta_3\gamma^2}, r \right\}.$$

Let  $Y$  be the closed ball in  $\mathbb{R}^n$  centered at 0, of radius  $\sigma$ . If  $\mu \in Y$ , in view of (2.6), we have

$$\lim_{\|x\|_1 \rightarrow +\infty} (Q(x) - \langle \psi(x), \mu \rangle_2) = +\infty$$

and so the function  $x \rightarrow Q(x) - \langle \psi(x), \mu \rangle_2$  is inf-compact. Further, observe that, by (c), the function  $\psi$  vanishes only at 0, while, by (a), 0 is not a global minimum of  $Q$ .

Clearly,  $Q$  is even and  $\psi$  is odd, in view of (c). In other words, each assumption of Theorem 2.2 is satisfied. Consequently, the number

$$\delta := \frac{1}{n} \left( \inf_{x \in X} \sup_{\mu \in Y} (Q(x) - \langle \psi(x), \mu \rangle_2) - \sup_{\mu \in Y} \inf_{x \in X} (Q(x) - \langle \psi(x), \mu \rangle_2) \right) \tag{2.7}$$

is positive. At this point, we apply Theorem 2.1 taking

$$h(x, \mu) = Q(x) - \langle \psi(x), \mu \rangle_2$$

for all  $(x, \mu) \in X \times Y$ . Fix  $n$  continuous functions  $g_1, \dots, g_n : \mathbb{R} \rightarrow \mathbb{R}$  satisfying

$$\max_{1 \leq k \leq n} \left( \sup_{t \in \mathbb{R}} \int_0^t g_k(s) ds - \inf_{t \in \mathbb{R}} \int_0^t g_k(s) ds \right) < \delta \tag{2.8}$$

and consider the function  $\varphi : X \rightarrow \mathbb{R}$  defined by

$$\varphi(x) = - \sum_{k=1}^n \int_0^{x_k} g_k(s) ds$$

for all  $x \in X$ . Clearly, in view of (2.7) and (2.8), we have

$$\begin{aligned} \sup_X \varphi - \inf_X \varphi &\leq \sum_{k=1}^n \left( \sup_{t \in \mathbb{R}} \int_0^t g_k(s) ds - \inf_{t \in \mathbb{R}} \int_0^t g_k(s) ds \right) \\ &\leq n \max_{1 \leq k \leq n} \left( \sup_{t \in \mathbb{R}} \int_0^t g_k(s) ds - \inf_{t \in \mathbb{R}} \int_0^t g_k(s) ds \right) \\ &< \inf_X \sup_Y h - \sup_Y \inf_X h. \end{aligned}$$

So, each assumption of Theorem 2.1 is satisfied. As a consequence, there exists  $\tilde{\mu} \in Y$  such that the function

$$J_{\tilde{\mu}}(\cdot) := h(\cdot, \tilde{\mu}) + \varphi(\cdot)$$

has at least two global minima in  $X$ . It is clear that this function  $J_{\tilde{\mu}}$  is  $C^1$ , with derivative given by

$$J'_{\tilde{\mu}}(x)(y) = K \left( \sum_{h=1}^{n+1} |x_h - x_{h-1}|^2 \right) \langle x, y \rangle_1 - \sum_{k=1}^n g_k(x_k) y_k - \sum_{k=1}^n \tilde{\mu}_k f_k(x_k) y_k$$

for all  $x, y \in X$ . So, taking into account that

$$\langle x, y \rangle_1 = - \sum_{k=1}^n (x_{k+1} - 2x_k + x_{k-1}) y_k,$$

we have

$$\begin{aligned}
 J'_{\tilde{\mu}}(x)(y) = & -K \left( \sum_{h=1}^{n+1} |x_h - x_{h-1}|^2 \right) \sum_{k=1}^n (x_{k+1} - 2x_k + x_{k-1})y_k \\
 & - \sum_{k=1}^n g_k(x_k)y_k - \sum_{k=1}^n \tilde{\mu}_k f_k(x_k)y_k
 \end{aligned} \tag{2.9}$$

for all  $x, y \in X$ . Since  $J_{\tilde{\mu}}$  is coercive and has at least two global minima, by a classical theorem of Courant, it possesses at least three critical points which, by (2.9), are three solutions of the problem. □

Here is a remarkable corollary of Theorem 2.3.

**Corollary 2.4.** *Let  $f_1, \dots, f_n : \mathbb{R} \rightarrow \mathbb{R}$  be  $n$  continuous functions satisfying conditions (b) and (c) of Theorem 2.3. Then, for each  $a, r > 0$  and  $b < 0$ , there exists a number  $\delta > 0$  with the following property: for every  $n$ -tuple of continuous functions  $g_1, \dots, g_n : \mathbb{R} \rightarrow \mathbb{R}$  satisfying*

$$\max_{1 \leq k \leq n} \left( \sup_{t \in \mathbb{R}} \int_0^t g_k(s) ds - \inf_{t \in \mathbb{R}} \int_0^t g_k(s) ds \right) < \delta,$$

there exists  $(\tilde{\mu}_1, \dots, \tilde{\mu}_n) \in \mathbb{R}^n$ , with  $\sum_{k=1}^n |\tilde{\mu}_k|^2 < r^2$ , such that the problem

$$\begin{cases} - \left( a \sum_{h=1}^{n+1} |x_h - x_{h-1}|^2 + b \right) (x_{k+1} - 2x_k + x_{k-1}) = g_k(x_k) + \tilde{\mu}_k f_k(x_k), & k=1, \dots, n, \\ x_0 = x_{n+1} = 0 \end{cases}$$

has at least three solutions.

*Proof.* It is enough to observe that the function  $K(t) = at + b$  satisfies condition (a) of Theorem 2.3. □

**Remark 2.5.** It is important to remark that the technique adopted in the proof Theorem 2.3 cannot be used to treat the non-discrete problem, keeping condition (a). This is due to the fact that, under condition (a), the functional

$$u \mapsto \int_0^{\int_{\Omega} |\nabla u(x)|^2 dx} K(s) ds$$

is not weakly lower semicontinuous in  $H_0^1(\Omega)$ .

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