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TOWARD WOJDA'S CONJECTURE ON DIGRAPH PACKING

Jerzy Konarski and Andrzej Żak

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Abstract. Given a positive integer $m \leq n/2$, Wojda conjectured in 1985 that if D_1 and D_2 are digraphs of order n such that $|A(D_1)| \leq n - m$ and $|A(D_2)| \leq 2n - \lfloor n/m \rfloor - 1$ then D_1 and D_2 pack. The cases when $m = 1$ or $m = n/2$ follow from known results. Here we prove the conjecture for $m \geq \sqrt{8n} + 418275$.

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1. INTRODUCTION

Two graphs of the same order, G_1 and G_2, are said to *pack*, if G_1 is a subgraph of the complement $\bar{G}_2$ of G_2, or, equivalently, G_2 is a subgraph of the complement $\bar{G}_1$ of G_1. Similarly, two digraphs of the same order, D_1 and D_2, are said to *pack*, if D_1 is a subgraph of the complement $\bar{D}_2$ of D_2, or D_2 is a subgraph of the complement $\bar{D}_1$ of D_1. The problem of packing graphs has been studied intensively. Much less is known about packing of digraphs. In particular the following theorem was proved by Benhocine, Veldman and Wojda in [1].

Theorem 1.1 ([1]). *If D_1 and D_2 are digraphs of order n with arc sets $A(D_1)$ and $A(D_2)$ such that $|A(D_1)| \cdot |A(D_2)| < n(n-1)$, then there is a packing of D_1 and D_2.*

An easy consequence of Theorem 1.1 is

Theorem 1.2 ([1]). *If D_1 and D_2 are digraphs of order n with $|A(D_1)| + |A(D_2)| \leq 2n - 2$ then there is a packing of D_1 and D_2.*

The following problem was posed by Wojda in 1985 [3]: for every n, k, $1 \leq k \leq n(n-1)$, determine the smallest number $f(n,k)$ such that there exist digraphs D_1 and D_2 with $|A(D_1)| = k$ and $|A(D_2)| = f(n,k)$ for which there is no packing. It is known that $f(n,1)=n(n-1)$, $f(n,2)=\binom{n}{2}$, $f(n,3)=n(n-1)-\lfloor n/2\rfloor$ for $n\geq 7$, $f(n,n-1)=n$, and $f(n,n)=n-1$ (see [1]).

Wojda conjectured what follows.

Conjecture 1.3 ([3])**.** *For every m satisfying $2\leq m\leq n/2$,*

$$f(n,n-m)=2n-\left\lfloor\frac{n}{m}\right\rfloor.$$

The case when $m=1$ folows from Theorem 1.1 or 1.2. Wojda and Woźniak claimed in [4] that Conjecture 1.3 is true for $m=2$ and $m=n/2$, but the proofs are very long and have not been published. In fact the case when $m=n/2$ follows from Theorem 1.1, too.

The following example given in [3] of two digraphs which do not pack, show that $f(n,n-m)\leq 2n-\left\lfloor\frac{n}{m}\right\rfloor$. Let D_1 be a digraph with vertices $v_1,\ldots,v_n$ and $n-m$ arcs such that $d_{D_1}(v_i)=d^-_{D_1}(v_i)\geq\lfloor(n-m)/m\rfloor$ for $i=1,\ldots,m$, $d^+_{D_1}(v_j)=1$ and $d^-_{D_1}(v_j)=0$ for $j\geq m+1$. Let D_2 be the digraph on n vertices and $2n-\left\lfloor\frac{n}{m}\right\rfloor$ arcs having a vertex w with $d^+_{D_2}(w)=n-1$ and $d^-_{D_2}(w)=n-\left\lfloor\frac{n}{m}\right\rfloor+1$. In order to prove the lower bound, one has to prove that a packing exists for every pair of digraphs F_1 and F_2 of order n such that $|A(F_1)|\leq n-m$ and $|A(F_2)|\leq 2n-\lfloor n/m\rfloor-1$.

Our main result is the following.

Theorem 1.4. *If $m\geq\sqrt{8n}+418275$, then*

$$f(n,n-m)=2n-\left\lfloor\frac{n}{m}\right\rfloor.$$

The notation is standard. For a digraph D, by $V(D)$ we denote the vertex set of D, and by $A(D)$ the arc set of D. The indegree of v is denoted $d^-_D(v)$ and its outdegree is denoted $d^+_D(v)$. The *total degree* (or simply degree) of v, denoted by $d_D(v)$, is defined by $d_D(v)=d^+_D(v)+d^-_D(v)$. A vertex with $d^-_D(v)=0$ is called a *source*, and a vertex with $d^+_D(v)=0$ is called a *sink*. We define $N^+(v)$ and $N^-(v)$ by

$$N^+(v)=\{u\in V(D):vu\in A(D)\}\text{ and }N^-(v)=\{u\in V(D):uv\in A(D)\}.$$

A digraph $D=(V,A)$ is said to be *complete* if both $uv\in A$ and $vu\in A$, for all $u,v\in V$. The complete digraph on n vertices is denoted by K^*_n. The graph $G(D)=(V,E)$, where $V=V(D)$ and $uv\in E$ if and only if uv or vu or both are in $A(D)$, is called the *underlying graph* of D. In the sequel we use the following equivalent definition of a packing. Let G_1 and G_2 be two graphs with $|V(G_1)|\leq|V(G_2)|=n$. We say that G_1 and G_2 *pack* (into K_n) if there is an injection $f:V(G_1)\to V(G_2)$ such that $f(u)f(v)\notin E(G_2)$ whenever $uv\in E(G_1)$. Similarly, for two digraphs D_1 and D_2 satisfying $|V(D_1)|\leq|V(D_2)|=n$, we say that D_1 and D_2 *pack* (into K^*_n) if there is an injection $f:V(D_1)\to V(D_2)$ such that $f(u)f(v)\notin A(D_2)$ whenever $uv\in A(D_1)$. Thus D_1 and D_2 pack if D_1 is contained in $\bar{D}_2$. Furthermore, D_1 and D_2 pack into K^*_n if their underlying graphs pack into K_n (note, however, that the opposite may be false).

2. PRELIMINARIES

The main tool in our proof is the following theorem, recently proved by Györi, Kostochka, McConvey and Yager.

Theorem 2.1 ([2]). *Let $C = 418275$. Let G_1 and G_2 be graphs of order n satisfying $\Delta(G_1) \le n-2$ and $\Delta(G_2) \le n-2$. If*

$$|E(G_1)| + |E(G_2)| + \max\{\Delta(G_1), \Delta(G_2)\} < 3n - C,$$

then G_1 and G_2 pack.

It is an old folklore result that a graph G of minimal degree d contains every tree T with $d+1$ vertices. This can be achieved simply by embedding vertices of T greedily one by one. Since at most d vertices of G are occupied at any point, there is always enough room to embed another vertex of the tree. What is more, the embedding of the first vertex of a tree is completely arbitrary. Since G contains T if and only if the complement $\bar{G}$ of G and T pack, we have the folowing folklore result.

Lemma 2.2. *Let G be a graph with maximum degree $n-1-d$ and T be a tree with $d+1$ vertices. Let $u \in V(G)$ and $v \in V(T)$. Then there exist a packing $f : V(T) \to V(G)$ of T and G such that $f(v) = u$.*

When we replace a tree by a forest having k components, then we can start by an arbitrary embedding of k vertices from different components.

Lemma 2.3. *Let G be a graph with maximum degree $n-1-d$ and F be a forest having k components and $d+1$ vertices. Let $u_1, \dots, u_k \in V(G)$ and $v_1, \dots, v_k \in V(F)$ such that v_i, $i = 1, \dots, k$, belong to different components of F. Then there exist a packing $f : V(F) \to V(G)$ of F and G such that $f(v_i) = u_i$, $i = 1, \dots, k$.*

3. PROOF OF THEOREM 1.4

Proof. Let $C = 418275$ and $m \ge \sqrt{8n} + C$. Then the following two inequalities are true:

$$m + \frac{n}{m} - \frac{9n}{m-1} \ge C + 3, \tag{3.1}$$

$$m + 1 \ge \frac{3n}{m-1} + 4. \tag{3.2}$$

We will show that if D_1, D_2 are digraphs of order n such that $|A(D_1)| \le n - m$ and $|A(D_2)| \le 2n - \left\lfloor \frac{n}{m} \right\rfloor - 1$ then D_1 and D_2 pack. Let $G_1 = G(D_1)$ and $G_2 = G(D_2)$ be the underlying graphs of D_1 and D_2, respectively. Clearly,

$$|E(G_i)| \le |A(D_i)|$$
$$d_{G_i}(v) \le d_{D_i}(v),$$

for $i = 1, 2$ and every $v \in V(D_i)$. Furthermore,we will show what follows.

Claim 3.1. *If G_1 and G_2 pack into K_n then D_1 and D_2 pack into K_n^*.*

We say that T is a *tree-component* of D_1 if T is a component of G_1, T is a tree in G_1 and T has no symmetric arcs in D_1. Since $|A(D_1)| \leq n - m$, D_1 has at least m tree-components. Let $w_1 \in V(D_1)$ with $d_{G_1}(w_1) = \Delta(G_1)$, and $w_2 \in V(D_2)$ with $d_{G_2}(w_2) = \Delta(G_2)$. We consider two cases.

Case 1. $d_{G_1}(w_1) = \max\{\Delta(G_1), \Delta(G_2)\}$.
Let $T_1, \ldots, T_{m-1}$, with $|V(T_1)| \leq |V(T_2)| \leq \cdots \leq |V(T_{m-1})|$, be tree-components of D_1 such that $w_1 \notin V(T_i)$, $i = 1, \ldots, m-1$. Clearly,

$$\sum_{i=1}^{k} |V(T_i)| \leq \frac{kn}{m-1}. \tag{3.3}$$

Since $|E(G_2)| < 2n$, $\delta(G_2) \leq 3$. Let $v_2 \in V(G_2)$ with $d_{G_2}(v_2) = \delta(G_2)$. Define $G_2' = G_2 - (N_{G_2}(v_2) \cup \{v_2\})$. Thus

$$|V(G_2')| \geq n - 4. \tag{3.4}$$

Let

$$F = \bigcup_{i=1}^{\delta(G_2)} T_i.$$

Since $\Delta(G_2) \leq d_{G_1}(w_1) \leq n - m - 1$, by (3.2), (3.3) and (3.4), we have

$$\Delta(G_2') \leq n - m - 1 \leq n - 3\frac{n}{m-1} - 4 \leq |V(G_2')| - |V(F)|.$$

Therefore, by Lemma 2.3, there exist a packing f_F of F and G_2' such that $N_{G_2}(v_2) \subseteq f_F(V(F))$. Let $G_1' = G_1 - (V(F) \cup \{w_1\})$ and $G_2' = G_2 - (f_F(V(F)) \cup \{v_2\})$. Note that

$$\begin{aligned}
&|E(G_1')| + |E(G_2')| + \max\{\Delta(G_1'), \Delta(G_2')\} \\
&\leq n - m - d_{G_1}(w_1) + 2n - \left\lfloor \frac{n}{m} \right\rfloor - 1 + \max\{\Delta(G_1'), \Delta(G_2')\} \\
&\leq 3n - m - \left\lfloor \frac{n}{m} \right\rfloor - 1 \leq 3n - \frac{9n}{m-1} - C - 3 \\
&\leq 3(n - |V(F)| - 1) - C
\end{aligned}$$

by (3.1) and (3.3). Since,

$$\Delta(G_1'), \Delta(G_2') \leq d_{G_1}(w_1) \leq n - m - 1 \leq \left(n - 1 - \frac{3n}{m-1}\right) - 3 \leq |V(G_1')| - 2,$$

G_1' and G_2' pack by Theorem 2.1. Let $f' : V(G_1') \to V(G_2')$ be a packing of G_1' and G_2'. Then f such that $f(w_1) = v_2$, $f(u) = f_F(u)$ for every $u \in V(F)$, and $f(u) = f'(u)$ for every $u \in V(G_1')$ is a packing of G_1 and G_2. Therefore, by Claim 3.1, f is a packing of D_1 and D_2, as well.

Case 2. $d_{G_1}(w_1) < \max\{\Delta(G_1), \Delta(G_2)\}$.
Let T be a smallest tree-component of D_1. Clearly

$$|V(T)| \leq \left\lfloor \frac{n}{m} \right\rfloor. \tag{3.5}$$

Claim 3.2. *If $d^-_{D_2}(w_2) = 0$ or $d^+_{D_2}(w_2) = 0$ then D_1 and D_2 pack.*

Proof of Claim 3.2. Without a loss of generality we assume that $d^+_{D_2}(w_2) = 0$. Since, $|A(T)| = |V(T)| - 1$, T has both, a sink and a source. Let s_1 be a source of T. Define $D'_1 = D_1 - s_1$ and $D'_2 = D_2 - w_2$ and let G'_1 and G'_2 be their underlying graphs. Note that

$$\begin{aligned}
&|E(G'_1)| + |E(G'_2)| + \max\{\Delta(G'_1), \Delta(G'_2)\} \\
&\leq n - m + 2n - \left\lfloor \frac{n}{m} \right\rfloor - 1 - d_{G_2}(w_2) + \max\{\Delta(G'_1), \Delta(G'_2)\} \\
&\leq 3n - m - \left\lfloor \frac{n}{m} \right\rfloor - 1 \leq 3(n-1) - C.
\end{aligned}$$

If $\Delta(G'_2) \leq n - 3$ then G'_1 and G'_2 pack by Theorem 2.1, and so, by Claim 3.1, D'_1 and D'_2 pack as well. Otherwise, $d_{G_2}(w_2) \geq n - 2$ and so

$$\begin{aligned}
|A(D'_1)| + |A(D'_2)| &\leq n - m + 2n - \left\lfloor \frac{n}{m} \right\rfloor - 1 - (n-2) \\
&\leq 2n - m - \left\lfloor \frac{n}{m} \right\rfloor + 1 \leq 2(n-1) - 2.
\end{aligned}$$

Hence D'_1 and D'_2 pack by Theorem 1.2.

Let $f' : V(D'_1) \to V(D'_2)$ be a packing of D'_1 and D'_2. Then f such that $f(s_1) = w_2$ and $f(u) = f'(u)$ for every $u \in V(D'_1)$ is a packing of D_1 and D_2. □

Now we will construct a packing f_T of T and D_2 such that $w_2 \in f_T(V(T))$. Let l_1 be a leaf of T. Without a loss of generality we assume that l_1 is a sink in T. Let v_1 be the neighbor of l_1. By Claim 3.2 we may assume that

$$d^+_{D_2}(w_2) \geq 1. \tag{3.6}$$

Claim 3.3. *There exist a packing f_T of T and D_2 such that $w_2 \in f_T(V(T))$.*

Proof of Claim 3.3. Suppose first that $d^-_{D_2}(w_2) \leq n - 2$. Thus, there exist $x_2 \in V(G_2) \setminus (N^-_{D_2}(w_2) \cup \{w_2\})$. Let $T' = T - l_1$ and $G' = G_2 - w_2$. If every $u_2 \in V(G')$ satisfies $d_{G'}(u_2) \leq n - \lfloor n/m \rfloor$ then

$$d_{G'}(u_2) \leq n - 2 - \left(\left\lfloor \frac{n}{m} \right\rfloor - 2 \right) \leq |V(G')| - 1 - |E(T')|, \tag{3.7}$$

and, by Lemma 2.2, there is a packing f' of T' and G' such that $f'(v_1) = x_2$. By the choice of x_2, f_T such that $f(l_1) = w_2$ and $f(u) = f'(u)$ for every $u \in T'$ is a packing of T and D_2.

So we may assume that $d_{G'}(u_2) \geq n - \lfloor \frac{n}{m} \rfloor + 1$ for some $u_2 \in V(G')$. Since

$$d^+_{D_2}(w_2) + d^-_{D_2}(w_2) + d_{G'}(u_2) \leq |A(D_2)| \leq 2n - \left\lfloor \frac{n}{m} \right\rfloor - 1,$$

by (3.6), we have that

$$d^-_{D_2}(w_2) \leq n - 3.$$

Therefore, there exist $y_2 \in V(G_2) \setminus (N^-_{D_2}(w_2) \cup \{u_2, w_2\})$. Let $G'' = G_2 - \{u_2, w_2\}$. Now, since every $v_2 \in V(G'')$ satisfies

$$\begin{aligned} d_{G''}(v_2) + 2(n - \left\lfloor \frac{n}{m} \right\rfloor + 1) - 1 &\leq d_{G''}(v_2) + d_{G_2}(u_2) + d_{G_2}(w_2) - 1 \\ &\leq |E(G_2)| \leq 2n - \left\lfloor \frac{n}{m} \right\rfloor - 1, \end{aligned}$$

we have

$$d_{G''}(v_2) \leq \left\lfloor \frac{n}{m} \right\rfloor - 2 \leq |V(G'')| - 1 - |E(T')|,$$

for every $v_2 \in V(G'')$. Hence, by Lemma 2.2, there is a packing f'' of T' and G'' such that $f''(v_1) = y_2$. Again, f_T such that $f_T(l_1) = w_2$ and $f_T(u) = f''(u)$ for every $u \in T'$ is a packing of T and D_2.

So we may assume that

$$d^-_{D_2}(w_2) = n - 1.$$

Then

$$1 \leq d^+_{D_2}(w_2) \leq n - \left\lfloor \frac{n}{m} \right\rfloor \tag{3.8}$$

and

$$d_{G_2}(u_2) \leq n - \left\lfloor \frac{n}{m} \right\rfloor \text{ for every } u_2 \neq w_2. \tag{3.9}$$

Let s_1 be a source in T. Let G' be a graph that arises from G_2 by removing the edges between w_2 and $V(G_2) \setminus N^+_{D_2}$. By (3.8) and (3.9),

$$\Delta(G') \leq n - \left\lfloor \frac{n}{m} \right\rfloor \leq n - 1 - |E(T)|.$$

Thus, by Lemma 2.2, there is a packing f_T of G' and T such that $f_T(s_1) = w_2$. By the choice of s_1, f_T is also a packing of T and D_2. □

By Claim 3.3, there is a packing f_T of T and D_2 such that $w_2 \in f_T(V(T))$. Let $D'_1 = D_1 - V(T)$ and $D'_2 = D_2 - f_T(V(T))$, and G'_1 and G'_2 be their underlying graphs. Note that

$$\begin{aligned} &|E(G'_1)| + |E(G'_2)| + \max\{\Delta(G'_1), \Delta(G'_2)\} \\ &\leq n - m - |V(T)| + 1 + 2n - \left\lfloor \frac{n}{m} \right\rfloor - 1 - d_{G_2}(w_2) + \max\{\Delta(G'_1), \Delta(G'_2)\} \\ &\leq 3n - m - |V(T)| - \left\lfloor \frac{n}{m} \right\rfloor < 3(n - |V(T)|) - C. \end{aligned}$$

Hence, if $d_{G_2}(w_2) \leq n - \lfloor \frac{n}{m} \rfloor - 2$, then G'_1 and G'_2 pack by Theorem 2.1. Thus, by Claim 3.1, D'_1 and D'_2 pack, as well. Otherwise, $d_{G_2}(w_2) \geq n - \lfloor \frac{n}{m} \rfloor + 1$, and so

$$\begin{aligned}|A(D'_1)| + |A(D'_2)| &\leq n - m - |V(T)| + 1 + 2n - \left\lfloor \frac{n}{m} \right\rfloor - 1 - d_{G_2}(w_2) \\ &\leq 3n - m - |V(T)| - \left\lfloor \frac{n}{m} \right\rfloor - \left(n - \left\lfloor \frac{n}{m} \right\rfloor + 1\right) < 2(n - |V(T)|) - 2.\end{aligned}$$

Thus, by Theorem 1.2, D'_1 and D'_2 pack.

Let f' be a packing of D'_1 and D'_2. Then f such that $f(u) = f'(u)$ for every $u \in V(D'_1)$ and $f(u) = f_T(u)$ for every $u \in V(T)$ is a packing of D_1 and D_2. $\square$

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Jerzy Konarski
konarski@agh.edu.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
al. A. Mickiewicza 30, 30-059 Krakow, Poland

Andrzej Żak
zakandrz@agh.edu.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
al. A. Mickiewicza 30, 30-059 Krakow, Poland

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