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ELASTO-PLASTIC MODEL OF UNSATURATED SOIL

1. Introduction

Amongst the many propositions of an elasto-plastic model of unsaturated soil media, extensions [1, 8, 11] of the concept of the classic model for Modified Cam-Clay [7] form an essential group. These concepts take into consideration the effect of liquid and gas phases on the stress state. They are based on an extension of the Critical State Theory, where suction is enclosed into three factors defining the state of the surface (effective values of shear and medium stresses and void ratio or specific volume).

The following paper presents such a model, constituting an actualized version of the proposition described in [3], transformed from the “net stress — strain and suction” system to the “effective stress — strain and suction” relation. The model uses a modified equation of the wheeler and Sivakumar’s yield surface [12] and the generalized hardening rule (connecting increment of plastic part of void ratio with stress and suction levels). This presentation is preceded by a derivation of the actualized extended constitutive elasto-plasticity rule [5] and it finishes with a discussion on the consideration of the suction effect.

2. Theoretical basis

In a three-phase medium soil the pores are partially filled with water and gas (in the form of bubbles in water). Suction s, defined as the difference between gas pressure ua and water pressure us (existing in pores), which is the effect of soil unsaturation. The stress state in any material point of such a medium is characterized by component σ'_{ij} of effective stress vector σ' defined by relationship [3, 5]

$$\sigma'_{ij} = (\sigma_{ij} - \delta_{ij} \cdot u_a) + \delta_{ij} \cdot \chi \cdot (u_a - u_w) = \sigma^n_{ij} + \delta_{ij} \cdot \chi \cdot s \tag{1}$$

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in which: σ_{ij} — is the component of the total stress vector σ, χ — is a parameter depending on saturation level of the soil, δ_{ij} — is Kronecker's delta. The relationship between the first pair of parenthesis is a component of the net stress vector $\boldsymbol{\sigma}^n$, defined as the difference between the component of total stress vector $\boldsymbol{\sigma}$ and gas pressure u_a. Standard invariant values — effective mean stress *p'* and total shear stress *q* — are connected with their net equivalents by the following formulae:

$$\begin{aligned} p' &= \frac{\sigma'_{11} + 2\cdot\sigma'_{33}}{3} = \frac{\sigma^n_{11} + 2\cdot\sigma^n_{33}}{3} + \chi\cdot s = p^n + \chi\cdot s \\ q &= q' = \sigma'_{11} - \sigma'_{33} = \sigma^n_{11} - \sigma^n_{33} = q^n \end{aligned} \tag{2}$$

3. Incremental equation of elasto-plasticity

The substitution of net values (p^n, q^n) along with stress invariants (*p'*, *q*) used to describe the stress state leads to a replacement of the extension elasticity rule [3–5] (taking suction *s* into account) with the classic version of the rule. As a result, the derivation of the constitutive equation of elasto-plasticity for a model of the unsaturated soil medium becomes necessary.

The stress state and strain state in the material point of the skeleton of the unsaturated soil medium is connected with a set of 6 basic relations: the rule of strain additivity, the constitutive relationship of elasticity, the plastic flow rule, consistency condition, the yield surface equation and the hardening rule:

$$\begin{aligned} &\text{a)} \quad d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^e + d\boldsymbol{\varepsilon}^p \\ &\text{b)} \quad d\boldsymbol{\sigma}' = \mathbf{D}\,d\boldsymbol{\varepsilon}^e \\ &\text{c)} \quad d\varepsilon^p = d\lambda\left\{\frac{\partial F}{\partial \boldsymbol{\sigma}'}\right\} = d\lambda\,\mathbf{a}_F \\ &\text{d)} \quad dF = \mathbf{a}_F^T d\boldsymbol{\sigma}' + \frac{\partial F}{\partial s}ds + \frac{\partial F}{\partial \eta}d\eta = 0 \\ &\text{e)} \quad F(p', q, s, \eta) = 0 \\ &\text{f)} \quad \eta = \eta(\boldsymbol{\varepsilon}^p, s) \end{aligned} \tag{3}$$

Quantities $d\boldsymbol{\varepsilon}^e$, $d\boldsymbol{\varepsilon}^p$ denote the increment vectors of the elastic and plastic parts of the strain increment $d\boldsymbol{\varepsilon}$, $d\boldsymbol{\sigma}'$ — is the vector of the effective stress increment, $\mathbf{D}$ — is the constitutive matrix of soil elasticity, ds — is the suction increment, $d\lambda$ — is the scalar multiplier, $\mathbf{a}_F = \{\partial F/\partial \boldsymbol{\sigma}'\}$ — is the gradient of plasticity function *F* (defined for values of effective mean stress *p'* and total shear stress *q*, suction *s* and scalar hardening parameter η, $\partial F/\partial s$, $\partial F/\partial \eta$ — are derivatives of yield function *F* for suction *s* and hardening parameter η.

The standard advantage described in [3], is realized in the analogical form with classical [6], comprises:

1) inclusion of the elastic part of the strain increment $d\boldsymbol{\varepsilon}^e$ from the transformed relation (3.a) into the elastic rule (3.b) and of the plastic increment $d\boldsymbol{\varepsilon}^p$ from the plastic flow rule (3.c):

$$d\boldsymbol{\sigma}' = \mathbf{D} \cdot (d\varepsilon - d\lambda \mathbf{a}_F) \tag{4}$$

2) inclusion of the obtained relation (4) for the increment of effective stress $d\boldsymbol{\sigma}'$ into the consistency condition (3.d):

$$\mathbf{a}_F^T \cdot \mathbf{D} \cdot (d\varepsilon - d\lambda \mathbf{a}_F) + \frac{\partial F}{\partial s} ds + \frac{\partial F}{\partial \eta} d\eta = 0 \tag{5}$$

3) the specification of the increment of the hardening parameter $d\eta$ through the differentiation of the hardening rule (3.f) and the consideration of the plastic flow rule (3.c):

$$d\eta = \left\{ \frac{\partial \eta}{\partial \varepsilon^p} \right\}^T d\varepsilon^p + \frac{\partial \eta}{\partial s} ds = \left\{ \frac{\partial \eta}{\partial \varepsilon^p} \right\}^T d\lambda \mathbf{a}_F + \frac{\partial \eta}{\partial s} ds \tag{6}$$

4) inclusion of the relation (6) for increment of the hardening parameter $d\eta$ into the modified consistency condition (5):

$$\mathbf{a}_F^T \cdot \mathbf{D} \cdot (d\varepsilon - d\lambda \mathbf{a}_F) + \frac{\partial F}{\partial s} ds + \frac{\partial F}{\partial \eta} \left(\left\{ \frac{\partial \eta}{\partial \varepsilon^p} \right\}^T d\lambda \mathbf{a}_F + \frac{\partial \eta}{\partial s} ds \right) = 0 \tag{7}$$

5) determination of scalar multiplier $d\lambda$ from the obtained expression (7):

$$d\lambda = \frac{\mathbf{a}_F^T \cdot \mathbf{D} \cdot d\varepsilon + \left(\frac{\partial F}{\partial s} + \frac{\partial F}{\partial \eta} \cdot \frac{\partial \eta}{\partial s} \right) ds}{\mathbf{a}_F^T \cdot \mathbf{D} \cdot \mathbf{a}_F - \frac{\partial F}{\partial \eta} \left\{ \frac{\partial \eta}{\partial \varepsilon^p} \right\}^T \cdot \mathbf{a}_F} \tag{8}$$

6) inclusion of the obtained formula (8) into relation (4) and ordering of this expression:

$$\begin{aligned} d\boldsymbol{\sigma}' &= \left[\mathbf{D} - \frac{(\mathbf{D} \cdot \mathbf{a}_F) \cdot (\mathbf{a}_F^T \cdot \mathbf{D})}{\mathbf{a}_F^T \cdot \mathbf{D} \cdot \mathbf{a}_F + K_F} \right] d\varepsilon + \frac{(\mathbf{D} \cdot \mathbf{a}_F) \cdot \left(\frac{\partial F}{\partial s} + \frac{\partial F}{\partial \eta} \cdot \frac{\partial \eta}{\partial s} \right)}{\mathbf{a}_F^T \cdot \mathbf{D} \cdot \mathbf{a}_F + K_F} ds \\ K_F &= -\frac{\partial F}{\partial \eta} \cdot \left\{ \frac{\partial \eta}{\partial \varepsilon^p} \right\}^T \cdot \mathbf{a}_F \end{aligned} \tag{9}$$

The obtained expression is the incremental constitutive equation of elasto-plastic partially saturated soil medium. K_F is the plastic hardening modulus. The relationship (9) consists of two parts, where the first part is identical to classic relation for elasto-plastic unsaturated soil medium. The second one considers effect of suction on stress state in soil skeleton.

4. Yield surface

Plastic behaviour of unsaturated soil medium is specified by the following pair of relationships: yield surface equation and hardening rule.

Yield surface [3] is defined by a modified Wheeler and Sivakumar's equation [12]. This expression is based on ellipse equation (fig. 1.a) in canonical form (captured on the plane "q'— p^n" for a specific level of suction s):

$$\begin{aligned} F(q,p^n,s) &= \frac{(q)^2}{b^2} + \frac{(p^n - d)^2}{a^2} - 1 = \\ &= \frac{(q)^2}{[M(s)\cdot p_x^n + \mu(s)]^2} + \frac{(p^n - p_x^n)^2}{(p_0^n - p_x^n)^2} - 1 = 0 \end{aligned} \tag{10}$$

Connected quantities p^n_x and p^n_0 denote values of net mean stresses determining position of top point C and origin point A for ellipse. $M(s)$ and $\mu(s)$ constitute a pair of model parameters defining the position of momentary critical state line $CSL(s)$ (current for suction level s). Inclusion of the transformed relation (2.a) into relationship (11) taking into account the correction of coordinates:

$$\begin{aligned} p^n &= p' - \chi\cdot s \\ p_x^n &= p'_x - \chi\cdot s \\ p_0^n &= p'_0 - \chi\cdot s \end{aligned} \tag{11}$$

gives final form of yield surface equation:

$$F(q,p',s) = \frac{(q)^2}{[M(s)\cdot(p'_x - \chi\cdot s) + \mu(s)]^2} + \frac{(p' - p'_x)^2}{(p'_0 - p'_x)^2} - 1 = 0 \tag{12}$$

The equation specifies yield surface on plane "q — p'" for specific suction level s.

The elastic state inside the ellipse, defined by rule (3.b) for invariant pairs of stress state (shear stress increment dq, effective mean stress increment dp') and strain state (shear strain increment $d\boldsymbol{\varepsilon}_q$, volumetric strain increment $d\boldsymbol{\varepsilon}_v$), is specified by the following relation:

$$\begin{Bmatrix} dq \\ dp' \end{Bmatrix} = \begin{bmatrix} 3\cdot G & 0 \\ 0 & K \end{bmatrix} \cdot \begin{Bmatrix} d\varepsilon_q \\ d\varepsilon_v \end{Bmatrix} \tag{13}$$

The elastic parameters G and K denote shear modulus and bulk modulus. When the non-linear elasticity is taken into consideration, the K modulus is changeable (variable depending on actual stress p', void ratio e and elastic swelling index κ):

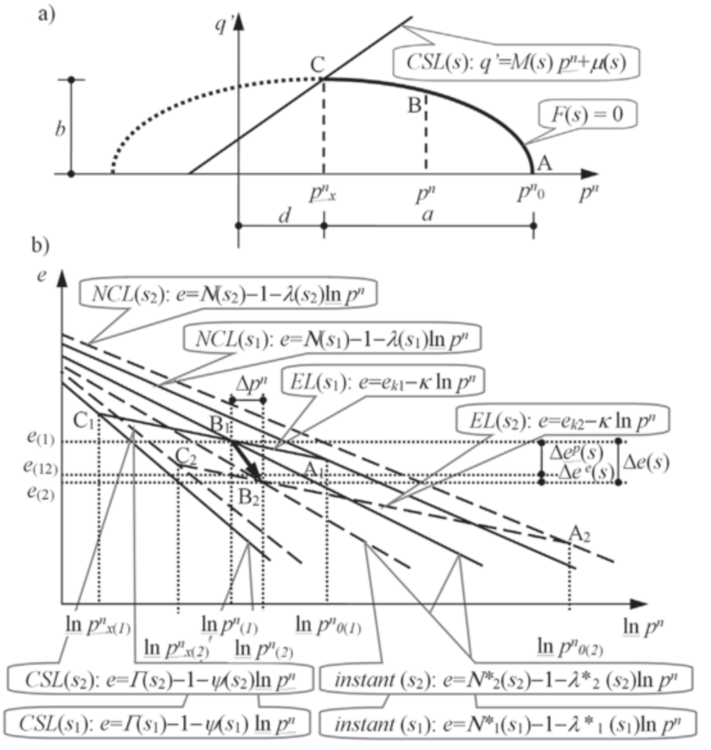

Fig. 1. Model scheme: a — yield surface F, b — consolidation lines for pass variant from stress level (1) to (2)

$$K = \frac{(1+e)}{\kappa} \cdot p' \tag{14}$$

5. Hardening rule

The hardening rule constitutes to the generalization of the classical concept for Modified Cam-Clay [7] in the difference version. The rule connects the plastic part of the void ratio $\Delta e^p(s)$ with the net stress state p^n. This rule is an effect of the analysis of 2 sets of consolidation characteristics (normal characteristic *NCL*, limit characteristic *CLS* and the elastic swelling *EL* marked "as function" for states (1) and (2)) in system "$e - \ln p^n$" (Fig. 1.b) for the pass variant of the stress state from level (1) to (2), corresponding with the suction change $\Delta s = s_2 - s_1$. The passing, illustrated by the vector connecting points B1 and B2, is accompanied by an adequate increase of the yield surface (assigned by coordinates of points A2 and C2 on the plane "$q' - p^n$").

The hardening rule is defined by expression for net mean stress $p^n_{(2)}$ [2]:

$$\begin{aligned}
&p^n_{(2)} = \sqrt[\beta]{(p^n_{(1)})^{\delta} \cdot \exp[\alpha - \Delta e^p \cdot (s)]} \\
&\alpha = [N(s_2) - N(s_1) - e_{k2}(s_2) + e_{k12}(s_1)] \\
&\delta = [\kappa - \lambda \cdot (s_1)] \\
&\beta = [\kappa - \lambda \cdot (s_2)]
\end{aligned} \tag{15}$$

Constants $N(s_2)$, $\lambda(s_2)$, $N(s_1)$, $\lambda(s_1)$ denote the coefficients of the indirect consolidation lines (shift ratios and inclination ratios respectively), $e_{k2}(s_2)$, $e_{k12}(s_2)$ — are shift ratios of the elastic swelling lines *EL*, κ — is an inclination ratio of line *EL*. This relation is complemented by a pair of expressions for net mean stress values $p^n{}_0$ and $p^n{}_x$, defining the correction of the position yield surface *F* in state (2) (i.e. coordinates of points A2 and C2 in system "$q' - p^n$"):

$$\begin{aligned}
&p^n_{0(2)} = \exp\left\{\frac{\Gamma(s_2) - \xi}{\psi(s_2) - \kappa}\right\} \\
&p^n_{x(2)} = \exp\left\{\frac{N_{(2)}(s_2) - \xi}{\lambda(s_2) - \kappa}\right\} \\
&\xi = 1 - e_{(2)}(s_2) + \kappa \cdot \ln p^n_{(2)}
\end{aligned} \tag{16}$$

Constants $\Gamma(s_2)$, $\psi(s_2)$, $N_{(2)}(s_2)$, $\lambda(s_2)$ denote coefficients of the consolidation lines *CS*L and *NCL* for state (2) (shift ratios and inclination ratios), $e_2(s_2)$ — is the void ratio for net mean stress $p^n{}_{(2)}$.

The inclusion of the transformed relation (2.a) into relationships (15)–(16) with consideration of signs given to stress levels (1) and (2):

$$\begin{aligned}
&p^n_{(2)} = p'_{(2)} - \chi \cdot s_2 \\
&p^n_{(1)} = p'_{(1)} - \chi \cdot s_1 \\
&p^n_{x(2)} = p'_{x(2)} - \chi \cdot s_2 \\
&p^n_{0(2)} = p'_{0(2)} - \chi \cdot s_2
\end{aligned} \tag{17}$$

gives the final form of the hardening rule and expressions for correcting the position of yield surface *F*:

$$\begin{aligned}
&p'_{(2)} = \sqrt[\beta]{(p'_{(1)} - \chi \cdot s_1)^{\delta} \cdot \exp[\alpha - \Delta e^p(s)]} + \chi \cdot s_2 \\
&p'_{0(2)} = \exp\left\{\frac{\Gamma(s_2) - \xi}{\psi(s_2) - \kappa}\right\} + \chi \cdot s_2 \\
&p'_{x(2)} = \exp\left\{\frac{N_{(2}(s_2) - \xi}{\lambda(s_2) - \kappa}\right\} + \chi \cdot s_2 \\
&\alpha = [N \cdot (s_2) - N \cdot (s_1) - e_{k2}(s_2) + e_{k12}(s_1)] \\
&\delta = [\kappa - \lambda \cdot (s_1)] \\
&\beta = [\kappa - \lambda \cdot (s_2)] \\
&\xi = 1 - e_{(2)}(s_2) + \kappa \cdot \ln(p'_{(2)} - \chi \cdot s_2)
\end{aligned} \tag{18}$$

6. Model parameters

The described model specifies:

1) is a set of 6 parametric relations $M(s)$, $\mu(s)$, $N(s)$, $\lambda(s)$, $\Gamma(s)$, $\psi(s)$, interpolating the results of the triaxial tests (for constant suction s levels) with using Lagrange polynomials, according to the formula:

$$y(s) = \sum_{j=1}^{n} y_j \prod_{\substack{i=1 \\ i \neq j}}^{n} \frac{s - s_i}{s_j - s_i} \tag{19}$$

where $y(s)$ denotes one of 6 relations $M(s)$, $\mu(s)$, $N(s)$, $\lambda(s)$, $\Gamma(s)$, $\psi(s)$, y_j — is the value of adequate parametric relation for suction s_j in point j.

2) is a set of 3 elastic parameters: the elastic swelling index κ, the shear modulus G and the bulk modulus K,
3) the 3 initial values: suction s_0, void ratio $e_0(s_0)$ and the effective mean stress $p'_0(s_0)$. Example values y_j have been contained within table 1.

TABLE 1
Nodal values of selected function parameters for kaolin [12]

i	s_i kPa]	$M_i = M(s_i)$	$\mu_i = \mu(s_i)$ [kPa]	$N_i = N(s_i)$	$\lambda_i = \lambda(s_i)$	$\Gamma_i = \Gamma(s_i)$	$\psi_i = \psi(s_i)$
1	0	0,813	0,0	2,052	0,128	2,011	0,110
2	100	0,933	54,2	2,122	0,182	1,984	0,108
3	200	0,959	83,5	2,196	0,196	2,042	0,181
4	300	0,910	122,0	2,212	0,176	2,105	0,223

7. Modelling of suction changes

The model presented of an unsaturated soil medium requires the specification of additional relationships, defining the effect of water and gas behaviour in pores on changes of pressures u_a and u_w, i.e. on change of suction s. Two approaches are applied here:

1) A complete model — using the Darcy law and the Fick law (for describing of water and gas behaviour) and giving the possibility to construct a complex variational equation of the balance for all phases of soil (used for Finite Element analysis of specific problems within the framework of the consolidation theory [2]),
2) A simplified model — using the experience of characteristics connecting suction with stress in the soil with the elementary physical parameters (degrees of saturation, void ratio and specific volume), taking into consideration the boundary-the initial conditions of the analyzed problems as additional relations [8].

Presently the second variant predominates in the applications and in the verification of the proposed models, because it allows one to limit research work to the analysis of the soil medium with the consideration of modeling the elasto-plastic properties. This concept makes it easier to use well known numerical techniques [8–10] for the numerical analysis of the specific but experimentally verified, stress paths.

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