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# Conditional Value-at-Risk and Value-at-Risk for Portfolio Optimization Model with Weighting Approach

## 1. Introduction

The portfolio problem involves computing the proportion of the initial budget that should be allocated in the available securities, is at the core of the field of financial management. A fundamental answer to this problem was given by Markowitz (1952 [4]) who proposed the mean-variance model which laid the basis of modern portfolio theory. Since the mean-variance theory of Markowitz, an enormous amount of papers have been published extending or modifying the basic model in three directions (e.g. Alexander and Baptista, 2004 [1]; Benati and Rizzi, 2007 [2]; Mansini *et al.*, 2007 [3]; Ogryczak W., 2000 [5]; Sawik, 2010 [9, 10], Sawik, 2009 [11, 12], 2008 [13]; Speranza, 1993 [14]). The proposed multi-criteria portfolio approach allows two percentile measures of risk in financial engineering: value-at-risk (VaR) and conditional value-at-risk (CVaR) to be applied for managing the risk of portfolio loss. The proposed mixed integer and linear programming models provide the decision maker with a simple tool for evaluating the relationship between expected and worst-case loss of portfolio return.

## 2. Definitions of Percentile Measures of Risk

VaR and CVaR have been widely used in financial engineering in the field of portfolio management (e.g. Sarykalin *et al.*, 2008 [8]). CVaR is used in conjunction with VaR and is applied for estimating the risk with non-symmetric cost distributions. Uryasev (2000 [15]) and Rockafellar and Uryasev (2000 [6], 2002 [7]) introduced a new approach to select a portfolio with the reduced risk of high losses. The portfolio is optimized by calculating VaR and minimizing CVaR simultaneously.

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Let $\alpha \in (0,1)$ be the confidence level. The percentile measures of risk, VaR and CVaR can be defined as below (see Fig. 1):

- Value-at-Risk (VaR) at a 100α% confidence level is the targeted return of the portfolio such that for 100α% of outcomes, the return will not be lower than VaR. In other words, VaR is a decision variable based on the α-percentile of return, i.e., in of 100(1–α)% outcomes, the return may not attain VaR.
- Conditional Value-at-Risk (CVaR) at a 100α% confidence level is the expected return of the portfolio in the worst 100(1–α)% of the cases. Allowing 100(1–α)% of the outcomes not exceed VaR, and the mean value of these outcomes is represented by CVaR.

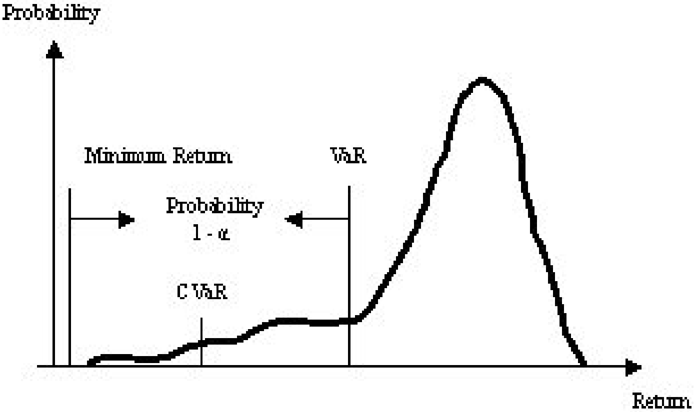


**Fig. 1.** Value-at-Risk and Conditional Value-at-Risk

## 3. Problem formulations

This section presents mathematical programming formulation for the multi-objective selection of optimal portfolio (for notation used, see Tab. 1). The presented portfolio model (Sawik, 2010 [9]) provides flexibility in how a decision maker wants to balance its risk tolerance with the expected portfolio returns. In the risk averse objective (1), VaR is a decision variable denoting the Value-at-risk. The nondominated solution set of the multi-objective portfolio can be found by the parametrization on λ. Constraint (2) ensures that all capital is invested in the portfolio (the selected securities). Risk constraint (3) defines the tail return for scenario *i*. Constraint (4) eliminates from selection the assets with non-positive expected return over all scenarios. Eq. (5) and (6) are non-negativity conditions.

Maximize

$$\lambda\left(VaR-(1-\alpha)^{-1}\sum_{i=1}^{m}p_iR_i\right)+(1-\lambda)\left(\sum_{i=1}^{m}p_i\left(\sum_{j=1}^{n}r_{ij}x_j\right)\right) \tag{1}$$

Subject to

$$\sum_{j=1}^{n} x_j = 1 \tag{2}$$

$$R_i \geq VaR - \sum_{j=1}^{n} r_{ij} x_j\,, \quad i \in M \tag{3}$$

$$x_j = 0, \quad j \in N: \ \sum_{i=1}^{m} p_i r_{ij} \leq 0 \tag{4}$$

$$x_j \geq 0, \quad j \in N \tag{5}$$

$$R_i \geq 0, \quad i \in M \tag{6}$$

**Table 1**
Notation

| |
|---|
| **Indices** |
| $i$ – historical time period $i$, $i \in M = \{1,\ldots,m\}$ (day) |
| $j$ – security $j$, $j \in N = \{1,\ldots,n\}$ |
| **Input Parameters** |
| $r_{ij}$ – observed return of security $j$ in historical time period $i$ |
| $p_i$ – probability of historical portfolio realization $i$ |
| $\alpha$ – confidence level |
| $\lambda$ – weight in the objective function |
| **Decision Variables** |
| $R_i$ – tail return, i.e., the amount by which $VaR$ exceeds returns in scenario $i$ |
| $VaR$ – Value-at-risk of portfolio return based on the $\alpha$-percentile of return, i.e., in $100\alpha\%$ of historical portfolio realization, the outcome must be greater than $VaR$ |
| $x_j$ – amount of capital invested in security $j$ |

## 4. Computational examples

In this section the strength of presented portfolio approach is demonstrated on computational examples (see Figs 2–5).

The data sets for the example problems were based on historic daily portfolios of the Warsaw Stock Exchange 4020 days (from 30th April 1991 to 30th of January 2009) and 240 securities. In the computational experiments the five levels of the confidence level was applied $\alpha \in \{0.99, 0.95, 0.90, 0.75, 0.50\}$, and for the weighted sum program the subset of nondominated solutions were computed by parametrization on $\lambda \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$.

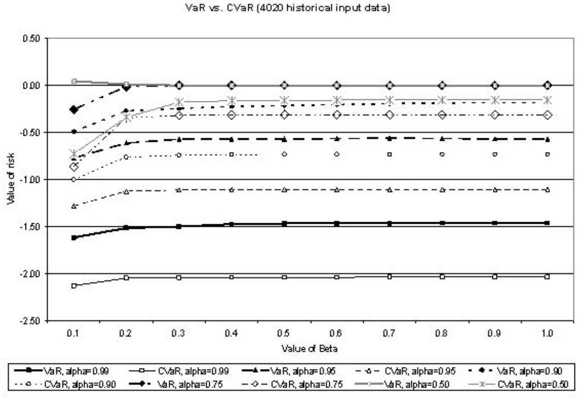


**Fig. 2.** CVaR vs. VaR

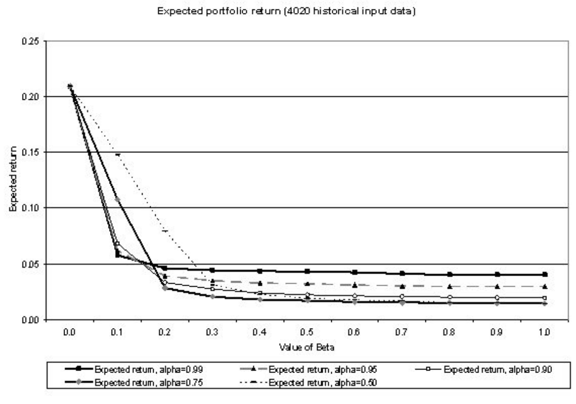


**Fig. 3.** Expected portfolio return

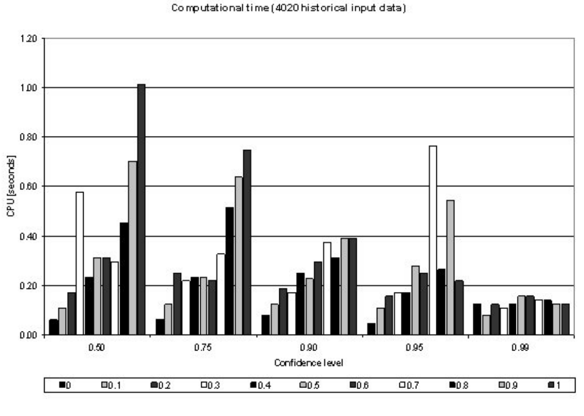


**Fig. 4.** Computational time

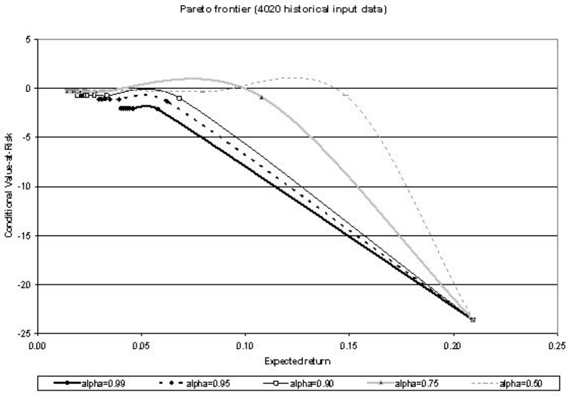


**Fig. 5.** Pareto frontier

All computational experiments were conducted on a laptop with Intel®Core 2 Duo T9300 processor running at 2.5 GHz and with 4GB RAM. For the implementation of the multi-objective portfolio models, the AMPL programming language and the CPLEX v.11 solver (with the default settings) were applied.

## 5. Conclusions

The portfolio approach presented in this paper has allowed the two popular in financial engineering percentile measures of risk, value-at-risk (VaR) and conditional value-at-risk (CVaR) to be applied. The computational experiments show that the proposed solution approach provides the decision maker with a simple tool for evaluating the relationship between the expected and the worst-case portfolio return. The decision maker can assess the value of portfolio return and the risk level, and can decide how to invest in a real life situation comparing with ideal (optimal) portfolio solutions. A risk-aversive decision maker wants to maximize the conditional value-at-risk. Since the amount by which losses in each scenario exceed VaR has been constrained of being positive, the presented models try to increase VaR and hence positively impact the objective functions. However, large increases in VaR may result in more historic portfolios (scenarios) with tail loss, counterbalancing this effect. The concave efficient frontiers illustrate the trade-off between the conditional value-at-risk and the expected return of the portfolio. In all cases the CPU time increases when the confidence level decreases. The number of securities selected for the optimal portfolio varies between 1 and more than 50 assets. Those numbers show very little dependence on the confidence level and the size of historical portfolio used as an input data. However, for the weighted-sum program, the weight parameter $\lambda$ does have an influence on the number of stocks selected for the optimal portfolio.

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