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THE ACHROMATIC NUMBER OF $K_6 \,\square\, K_7$ IS 18

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Abstract. A vertex colouring $f : V(G) \to C$ of a graph G is complete if for any two distinct colours $c_1, c_2 \in C$ there is an edge $\{v_1, v_2\} \in E(G)$ such that $f(v_i) = c_i$, $i = 1, 2$. The achromatic number of G is the maximum number $\operatorname{achr}(G)$ of colours in a proper complete vertex colouring of G. In the paper it is proved that $\operatorname{achr}(K_6 \,\square\, K_7) = 18$. This result finalises the determination of $\operatorname{achr}(K_6 \,\square\, K_q)$.

Keywords: complete vertex colouring, achromatic number, Cartesian product.

Mathematics Subject Classification: 05C15.

1. INTRODUCTION

Consider a finite simple graph G and a finite colour set C. A vertex colouring $f : V(G) \to C$ is *complete* if for any two distinct colours $c_1, c_2 \in C$ one can find an edge $\{v_1, v_2\} \in E(G)$ ($\{v_1, v_2\}$ is usually shortened to v_1v_2) such that $f(v_i) = c_i$, $i = 1, 2$. The *achromatic number* of G, in symbols $\operatorname{achr}(G)$, is the maximum cardinality of C admitting a proper complete vertex colouring of G. The invariant was introduced by Harary, Hedetniemi, and Prins in [4], where the following interpolation theorem was proved.

Theorem 1.1. *If G is a graph, and $\chi(G) \le k \le \operatorname{achr}(G)$ for an integer k, then there exists a k-element colour set C and a proper complete vertex colouring $f : V(G) \to C$.*

Let $G \,\square\, H$ denote the Cartesian product of graphs G and H (the notation follows the monograph [10] by Imrich and Klavžar). So, $V(K_p \,\square\, K_q) = V(K_p) \times V(K_q)$, and $E(K_p \,\square\, K_q)$ consists of all edges $(x, y_1)(x, y_2)$ with $y_1 \ne y_2$ and all edges $(x_1, y)(x_2, y)$ with $x_1 \ne x_2$. The problem of determining $\operatorname{achr}(K_p \,\square\, K_q)$ is motivated by the fact that, according to Chiang and Fu [2],

$$\operatorname{achr}(G \,\square\, H) \ge \operatorname{achr}(K_p \,\square\, K_q)$$

for arbitrary graphs G, H with $\operatorname{achr}(G) = p$ and $\operatorname{achr}(H) = q$. The graph $K_q \,\square\, K_p$ is isomorphic to the graph $K_p \,\square\, K_q$, hence

$$\operatorname{achr}(K_q \,\square\, K_p) = \operatorname{achr}(K_p \,\square\, K_q),$$

and so it is natural to suppose $p \leq q$. The case $p \in \{2, 3, 4\}$ was solved by Horňák and Puntigán in [9], and that of $p = 5$ by Horňák and Pčola in [7, 8].

Proposition 1.2 ([1])**.** $\operatorname{achr}(K_6 \,\square\, K_6) = 18$.

More generally, in [3] Chiang and Fu proved that if r is an odd projective plane order, then

$$\operatorname{achr}(K_{(r^2+r)/2} \,\square\, K_{(r^2+r)/2}) = (r^3 + r^2)/2.$$

The aim of the present paper is to finalise the determination of $\operatorname{achr}(K_6 \,\square\, K_q)$ (for $q \geq 8$ see Horňák [5, 6]) by proving

Theorem 1.3. $\operatorname{achr}(K_6 \,\square\, K_7) = 18$.

To formulate the complete result describing the achromatic number of $K_6 \,\square\, K_q$ we use the sets J_a, $a \in [3, 6]$, where

$$\begin{aligned} J_3 &= [2,3] \cup \{q \in [41, \infty) : q \equiv 1 (\mathrm{mod}\ 2)\}, \\ J_4 &= \{1, 4, 7\} \cup [16, 40] \cup \{q \in [42, \infty) : q \equiv 0 (\mathrm{mod}\ 2)\}, \\ J_5 &= \{5, 8\}, \\ J_6 &= \{6\} \cup [9, 15]. \end{aligned}$$

Note that $J_3 \cup J_4 \cup J_5 \cup J_6 = [1, \infty)$.

Theorem 1.4. *If* $a \in [3, 6]$ *and* $q \in J_a$*, then* $\operatorname{achr}(K_6 \square K_q) = 2q + a$.

2. NOTATION AND BASIC FACTS

For $k, l \in \mathbb{Z}$ we denote *integer intervals* by

$$[k, l] = \{z \in \mathbb{Z} : k \leq z \leq l\}, \quad [k, \infty) = \{z \in \mathbb{Z} : k \leq z\}.$$

Further, for a set A and $m \in [0, \infty)$ let $\binom{A}{m}$ be the set of m-element subsets of A.

Under the assumption that $V(K_r) = [1, r]$ for $r \in [1, \infty)$, a vertex colouring $f : V(K_p \,\square\, K_q) \to C$ can be conveniently described using the $p \times q$ matrix $M = M(f)$, in which the entry in the ith row and the jth column is $(M)_{i,j} = f(i, j)$.

If f is proper, then each line (row or column) of M consists of pairwise distinct entries.

If f is complete, then each pair $\{\gamma_1, \gamma_2\} \in \binom{C}{2}$ (of colours in C) is *good* in M, which means that at least one of the next two conditions is fulfilled:

(i) the pair $\{\gamma_1, \gamma_2\}$ is *row-based* (in M), i.e., there are $i \in [1, p]$ and $j_1, j_2 \in [1, q]$ such that $\{\gamma_1, \gamma_2\} = \{(M)_{i,j_1}, (M)_{i,j_2}\}$,

(ii) the pair $\{\gamma_1, \gamma_2\}$ is *column-based* (in M), i.e., there are $i_1, i_2 \in [1, p]$ and $j \in [1, q]$ such that $\{\gamma_1, \gamma_2\} = \{(M)_{i_1,j}, (M)_{i_2,j}\}$.

The following is a natural necessary condition for the completeness of f: Given $\gamma \in C$ and $C' \subseteq C \setminus \{\gamma\}$, the number $g(\gamma, C')$ of pairs $\{\gamma, \gamma'\}$ with $\gamma' \in C' \setminus \{\gamma\}$ that are good in M, is at least $|C'| - 1$. Note that

$$g(\gamma, C') \leq \sum_{(i,j):(M)_{i,j}=\gamma} g(i, j, C'),$$

where $g(i, j, C')$ is the number of those pairs $\{\gamma, \gamma'\}$, $\gamma' \in C' \setminus \{\gamma\}$, that are good in M due to the copy $(M)_{i,j}$ of the colour γ.

Let $\mathcal{M}(p, q, C)$ be the set of $p \times q$ matrices M with entries from C such that entries of any line of M are pairwise distinct, and all pairs in $\binom{C}{2}$ are good in M. Thus if $f : [1, p] \times [1, q] \to C$ is a proper complete vertex colouring of $K_p \,\square\, K_q$, then $M(f) \in \mathcal{M}(p, q, C)$.

Conversely, if $M \in \mathcal{M}(p, q, C)$, it is immediate to see that the mapping $f_M : [1, p] \times [1, q] \to C$ determined by $f_M(i, j) = (M)_{i,j}$ is a proper complete vertex colouring of $K_p \,\square\, K_q$.

So, we have just proved

Proposition 2.1. *If $p, q \in [1, \infty)$ and C is a finite set, then the following statements are equivalent:*

(1) *there is a proper complete vertex colouring of $K_p \,\square\, K_q$ using as colours elements of C,*

(2) $\mathcal{M}(p, q, C) \neq \emptyset$.

The following straightforward proposition comes from [5].

Proposition 2.2. *If $p, q \in [1, \infty)$, C, D are finite sets, $M \in \mathcal{M}(p, q, C)$, mappings $\rho : [1, p] \to [1, p]$, $\sigma : [1, q] \to [1, q]$, $\pi : C \to D$ are bijections, and $M_{\rho,\sigma}$, M_π are $p \times q$ matrices defined by $(M_{\rho,\sigma})_{i,j} = (M)_{\rho(i),\sigma(j)}$ and $(M_\pi)_{i,j} = \pi((M)_{i,j})$, then $M_{\rho,\sigma} \in \mathcal{M}(p, q, C)$ and $M_\pi \in \mathcal{M}(p, q, D)$.* $\square$

Let $M \in \mathcal{M}(p, q, C)$ and let $\gamma \in C$. For a colour $\gamma \in C$ and the colouring f_M from the proof of Proposition 2.1 denote $V_\gamma = f_M^{-1}(\gamma) \subseteq [1, p] \times [1, q]$, and let $N(V_\gamma)$ be the neighbourhood of V_γ (the union of neighbourhoods of vertices in V_γ). The *excess* of γ is defined to be the maximum number $\mathrm{exc}(\gamma)$ of vertices in a set $S \subseteq N(V_\gamma)$ such that the partial vertex colouring of $K_6 \,\square\, K_7$, obtained by removing colours of S, is still complete concerning the colour class γ.

The *frequency* of the colour γ is the number of entries of M equal to γ. An *l-colour* (*l+colour*) is a colour of frequency l (at least l), and C_l (C_{l+}) is the set of l-colours (l+colours). Further, for $k \in \{l, l+\}$ let $c_k = |C_k|$,

$$\mathbb{R}(i) = \{(M)_{i,j} : j \in [1, q]\}, \ \mathbb{R}_k(i) = C_k \cap \mathbb{R}(i), \ r_k(i) = |\mathbb{R}_k(i)|, \ i \in [1, p],$$
$$\mathbb{C}(j) = \{(M)_{i,j} : i \in [1, p]\}, \ \mathbb{C}_k(j) = C_k \cap \mathbb{C}(j), \ c_k(j) = |\mathbb{C}_k(j), \ j \in [1, q].$$

Finally, denote

$$\mathbb{R}_2(i_1, i_2) = C_2 \cap \mathbb{R}(i_1) \cap \mathbb{R}(i_2), \ r_2(i_1, i_2) = |\mathbb{R}_2(i_1, i_2)|, \ i_1, i_2 \in [1, p], i_1 \neq i_2,$$
$$\mathbb{C}_2(j_1, j_2) = C_2 \cap \mathbb{C}(j_1) \cap \mathbb{C}(j_2), \ c_2(j_1, j_2) = |\mathbb{C}_2(j_1, j_2)|, \ j_1, j_2 \in [1, q], j_1 \neq j_2.$$

Considering a nonempty set $S \subseteq [1,p] \times [1,q]$ we say that a colour $\gamma \in C$ *occupies a position in* S (*appears in* S, *has a copy in* S or simply *is in* S) if there is $(i,j) \in S$ such that $(M)_{i,j} = \gamma$.

3. PROPERTIES OF A COUNTEREXAMPLE TO THEOREM 1.3

We prove Theorem 1.3 by the way of contradiction. It is well known that

$$\operatorname{achr}(G) \geq \operatorname{achr}(H)$$

if H is an induced subgraph of a graph G. So, by Proposition 1.2,

$$\operatorname{achr}(K_6 \,\square\, K_7) \geq \operatorname{achr}(K_6 \,\square\, K_6) = 18.$$

Provided that Theorem 1.3 is false, by Theorem 1.1 and Proposition 2.1 there is a set C with $|C| = 19$ and a matrix $M \in \mathcal{M}(6,7,C)$; henceforth the whole notation corresponds to this (hypothetical) matrix M.

Claim 3.1. *If* $\gamma \in C_l$, *then* $\operatorname{exc}(\gamma) = -l^2 + 12l - 18$.

Proof. The vertex colouring f_M of $K_6 \,\square\, K_7$ is proper, hence

$$|N(V_\gamma)| = 7l + l(6-l) - l = l(12-l).$$

Further, f_M is complete, and so each colour of $C \setminus \{\gamma\}$ appears on a vertex belonging to $N(V_\gamma)$. Therefore,

$$\operatorname{exc}(\gamma) = l(12-l) - (19-1) = -l^2 + 12l - 18. \qquad \square$$

Claim 3.2. *The following statements are true:*
1. $c_1 = 0$,
2. *if* $l \in [7,\infty)$, *then* $c_l = 0$,
3. $c_2 \in [15,18]$,
4. $c_{3+} \in [1,4]$,
5. $\Sigma = \sum_{i=3}^{6} ic_i \in [6,12]$,
6. $c_{4+} \leq c_2 - 15$,
7. *if* $c_{4+} = 0$, *then* $c_{3+} = c_3 = 4$,
8. *if* $c_{4+} \geq 1$, *then* $c_{3+} \leq 3$,
9. $c_{3+} + c_{4+} \leq 4$,
10. *if* $c_{5+} \geq 1$, *then* $c_{3+} + c_{4+} \leq 3$.

Proof. 1. If $\gamma \in C_1$, then, by Claim 3.1, $\operatorname{exc}(\gamma) = -7 < 0$, a contradiction.

2. If $\gamma \in C_l$ for some $l \in [7,\infty)$, then by the pigeonhole principle the colouring f_M is not proper, a contradiction.

3. By Claims 3.2.1, 3.2.2 we have

$$c_2 + c_{3+} = \sum_{i=2}^{6} c_i = |C| = 19$$

and

$$\sum_{i=2}^{6} ic_i = |V(K_6 \,\square\, K_7)| = 42,$$

hence

$$2c_2 + 6(19 - c_2) = 2c_2 + 6c_{3+} \geq \sum_{i=2}^{6} ic_i \geq 2c_2 + 3c_{3+} = 2c_2 + 3(19 - c_2),$$

which yields

$$114 - 4c_2 \geq 42 \geq 57 - c_2 \quad \text{and} \quad 15 \leq c_2 \leq 18.$$

4. A consequence of Claim 3.2.3.
5. The assertion of Claim 3.2.3 leads to

$$\sum_{i=3}^{6} ic_i = \sum_{i=2}^{6} ic_i - 2c_2 = 42 - 2c_2 \in [6, 12].$$

6. We have

$$3 \cdot 19 - c_2 + c_{4+} = 3(c_2 + c_3 + c_{4+}) - c_2 + c_{4+} \leq \sum_{i=2}^{6} ic_i = 42$$

and

$$c_{4+} \leq c_2 - 15.$$

7. If $c_{4+} = 0$, then

$$c_2 + c_3 = 19, \quad 2c_2 + 3c_3 = 42 \quad \text{and} \quad c_{3+} = c_3 = 42 - 2 \cdot 19 = 4.$$

8. The assumption $c_{4+} \geq 1$ and $c_{3+} = 4$ would mean

$$\Sigma \geq 3 \cdot 3 + 4 = 13 > 12,$$

which contradicts Claim 3.2.5.

9. If $c_{4+} = 0$, then $c_{3+} + c_{4+} = c_{3+} \leq 4$. With $c_{4+} = 1$ we have, by Claim 3.2.6,

$$\begin{aligned} &1 = c_{4+} \leq c_2 - 15, \quad c_2 \geq 16, \quad c_3 = 19 - c_2 - c_{4+} \leq 2, \\ &c_{3+} \leq 3 \quad \text{and} \quad c_{3+} + c_{4+} \leq 4. \end{aligned}$$

Finally, from $c_{4+} \geq 2$ it follows that

$$\begin{aligned} &2 \leq c_{4+} \leq c_2 - 15, \quad c_2 \geq 17, \\ &19 = c_2 + c_3 + c_{4+} \geq 17 + c_3 + 2 = 19 + c_3 \geq 19, \\ &c_2 = 17, \quad c_3 = 0, \quad c_{4+} = 2 = c_{3+} \quad \text{and} \quad c_{3+} + c_{4+} = 4. \end{aligned}$$

10. We have

$$2(19 - c_{5+}) + 5c_{5+} = 2(c_2 + c_3 + c_4) + 5(c_5 + c_6) \leq \sum_{i=2}^{6} ic_i = 42,$$
$$3c_{5+} \leq 4 \quad \text{and} \quad c_3 + 2c_4 = 42 - 2(19 - c_5 - c_6) - 5c_5 - 6c_6 \leq 4 - 3c_{5+},$$

hence $c_{5+} \geq 1$ yields

$$c_{5+} = 1, \quad c_3 + 2c_4 \leq 1, \quad c_{4+} = c_{5+} = 1,$$
$$c_{3+} = c_3 + c_{4+} = c_3 + 1 \quad \text{and} \quad c_{3+} + c_{4+} = c_3 + 1 + 1 \leq 3. \qquad \square$$

A set $D \subseteq C_2$ is of the *type* $(m_1^{a_1} \dots m_k^{a_k}, n_1^{b_1} \dots n_l^{b_l})$ if both $(m_1, \dots, m_k)$, $(n_1, \dots, n_l)$ are decreasing sequences of integers from the interval $[1, |D|]$, the number of rows of M containing m_i colours from D is $a_i \geq 1$ for each $i \in [1, k]$, the number of columns of M containing n_i colours from D is $b_i \geq 1$ for each $i \in [1, l]$, and

$$\sum_{i=1}^{k} m_i a_i = 2|D| = \sum_{i=1}^{l} n_i b_i.$$

Forthcoming Claims 3.3, 3.4, 3.6, 3.7, and 3.9 state that certain types of 2- and 3-element subsets of C_2 are impossible in a matrix M contradicting Theorem 1.3. The mentioned claims are proved by contradiction. When proving that M avoids a type T, we suppose that there is $D \subseteq C_2$, which is of the type T (without explicitly mentioning it), and we reach a contradiction with some of the properties following from the fact that $M \in \mathcal{M}(6, 7, C)$.

Claim 3.3. *No set* $\{\alpha, \beta\} \subseteq C_2$ *is of the type* $(1^4, 2^2)$.

Proof. Since we have at our disposal Proposition 2.2, we may suppose without loss of generality that $(M)_{1,1} = \alpha = (M)_{3,2}$ and $(M)_{2,1} = \beta = (M)_{4,2}$. We use (w) to express briefly that it is Proposition 2.2, which enables us to simplify our reasoning by restricting our attention to matrices with a special structure. With $A = \mathbb{C}(1) \cup \mathbb{C}(2)$ we have $|A| \leq 10$. If $\gamma \in C \setminus A$, then the fact that both pairs $\{\gamma, \alpha\}$ and $\{\gamma, \beta\}$ are good forces γ to occupy a position in $S_\alpha = \{1, 3\} \times [3, 7]$ and in $S_\beta = \{2, 4\} \times [3, 7]$ as well. So,

$$|C \setminus A| \leq 10 \quad \text{and} \quad |C \setminus A| = |C| - |A| \geq 9.$$

By Claim 3.2.4,

$$|(C \setminus A) \cap C_2| = |C \setminus A| - |(C \setminus A) \cap C_{3+}| \geq 9 - c_{3+} \geq 5,$$

hence there is $\delta \in (C \setminus A) \cap C_2$. Now, as δ is in both S_α and S_β, there is $(i, k) \in \{1, 3\} \times \{2, 4\}$ such that $\delta \in \mathbb{R}_2(i, k)$. If $(i, k) = (1, 2)$, then (w) $(M)_{1,3} = \delta = (M)_{2,4}$ (see Figure 1).

$$\begin{pmatrix} \alpha & . & \delta & \bullet & \bullet & \bullet & \bullet \\ \beta & . & \bullet & \delta & \bullet & \bullet & \bullet \\ . & \alpha & \bullet & \bullet & . & . & . \\ . & \beta & \bullet & \bullet & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix}$$

Fig. 1.

If $|C \setminus A| = 10$, then all ten positions in both S_α, S_β are occupied by colours of $C \setminus A$, and all twelve bullet positions in Figure 1 are occupied by colours of $(C \setminus A) \setminus \{\delta\}$, which means that

$$\operatorname{exc}(\delta) \geq 12 - |(C \setminus A) \setminus \{\delta\}| = 12 - (10 - 1) = 3$$

in contradiction to Claim 3.1.

If $|C \setminus A| = 9$, then $\mathbb{C}(1) \cap \mathbb{C}(2) = \{\alpha, \beta\}$. Let B be the set of four colours occupying a position in $[5,6] \times [1,2]$. Using $\operatorname{exc}(\alpha) = \operatorname{exc}(\beta) = 2$ we see that at most two positions in $[1,4] \times [3,7]$ are occupied by a colour of B. Thus, if $\varepsilon \in B$ is not in $[1,4] \times [3,7]$ (and there are at least two possibilities for such ε), then it must be in $[5,6] \times [3,7]$, and so $\varepsilon \in C_2$ (the colouring f_M is proper). Then, however, the number of pairs $\{\zeta, \varepsilon\}$ with $\zeta \in (C \setminus A) \cap C_2$ that are good is at most four (ζ must share the column with the copy of ε appearing in $[5,6] \times [3,7]$), while

$$|(C \setminus A) \cap C_2| \geq |C \setminus A| - |C_{3+}| \geq 9 - 4 = 5,$$

a contradiction.

Provided that $(i,k) \neq (1,2)$, a contradiction can be reached in a similar manner. □

Claim 3.4. *No set* $\{\alpha, \beta\} \subseteq C_2$ *is of the type* $(2^1 1^2, 2^2)$.

Proof. Now (w) $(M)_{1,1} = \alpha = (M)_{2,2}$ and $(M)_{2,1} = \beta = (M)_{3,2}$. With $A = \mathbb{C}(1) \cup \mathbb{C}(2) \cup \mathbb{R}(2)$ each colour $\gamma \in C \setminus A$ has a copy in $\{i\} \times [3,7]$, $i = 1, 3$ ($\{\gamma, \alpha\}$ and $\{\gamma, \beta\}$ are good). From $|A| \leq 15$ it follows that

$$|C \setminus A| \geq 19 - 15 = 4,$$

and then $C \setminus A \subseteq C_{3+}$: indeed, if $\delta \in (C \setminus A) \cap C_2$, then

$$\operatorname{exc}(\delta) \geq |(C \setminus A) \setminus \{\delta\}| \geq 3,$$

a contradiction. Thus $C_2 \subseteq A$, $c_2 \leq 15$, hence, by Claims 3.2.3, 3.2.4, $c_2 = 15$, $c_3 = c_{3+} = 4$, $C \setminus A = C_{3+} = C_3$, each colour of $C_2 \setminus \{\alpha, \beta\}$ has exactly one copy in $([1,6] \times [1,2]) \cup (\{2\} \times [3,7])$, and (w) $\varepsilon = (M)_{1,2}$, $\zeta = (M)_{3,1}$ are (distinct) 2-colours.

First note that $\varepsilon, \zeta \notin \mathbb{R}_2(1,3)$, for otherwise

$$\max(\operatorname{exc}(\varepsilon), \operatorname{exc}(\zeta)) \geq c_3 = 4.$$

So, the second copies of ε, ζ are in $[4,6] \times [3,7]$, and the pair $\{\varepsilon, \zeta\}$ is good in the corresponding 3×5 submatrix of M.

If the pair $\{\varepsilon, \zeta\}$ is column-based, (w) $\varepsilon = (M)_{4,3}$ and $\zeta = (M)_{5,3}$, then, with a (2-) colour η appearing in $\{2\} \times [4,7]$, both pairs $\{\eta, \varepsilon\}$ and $\{\eta, \zeta\}$ are good only if η occupies a position in $\{1,3,6\} \times \{3\}$, a contradiction.

If the pair $\{\varepsilon, \zeta\}$ is row-based, then (w) $\varepsilon = (M)_{4,3}$ and $\zeta = (M)_{4,4}$; consider six positions in $[5,6] \times [5,7]$. Since $r_3(1) = r_3(3) = 4 = c_3$, at most four of those positions are occupied by 3-colours and at least two of them by 2-colours. Let B be the set of 2-colours having a copy in $([5,6] \times [5,7]) \cup (\{2\} \times [5,7])$. If $\vartheta \in B$, then, having in mind that both pairs $\{\vartheta, \varepsilon\}$ and $\{\vartheta, \zeta\}$ are good, ϑ must have a copy in $\{(1,4),(3,3),(4,1),(4,2)\}$; this contradicts the inequality $|B| \geq 5$. ☐

Claim 3.5. *If $j, l \in [1,7]$, $j \neq l$, then $c_2(j,l) \leq 2$.*

Proof. The assumption $c_2(j,l) \geq 3$ would contradict Claim 3.3 or Claim 3.4. ☐

Claim 3.6. *No set $\{\alpha, \beta\} \subseteq C_2$ is of the type $(2^2, 1^4)$.*

Proof. Here (w) $(M)_{1,1} = \alpha = (M)_{2,3}$ and $(M)_{1,2} = \beta = (M)_{2,4}$. With $A = \mathbb{R}(1) \cup \mathbb{R}(2)$ we have $|A| \leq 12$, each colour of $C \setminus A$ is in both sets $S_\alpha = [3,6] \times \{1,3\}$, $S_\beta = [3,6] \times \{2,4\}$, and $7 \leq |C \setminus A| \leq 8$. As $|(C \setminus A) \cap C_2| \geq 3$, there is $(j,l) \in \{1,3\} \times \{2,4\}$ such that $\gamma \in (C \setminus A) \cap \mathbb{C}_2(j,l)$.

If $(j,l) = (1,2)$, then (w) $(M)_{3,1} = \gamma = (M)_{4,2}$ (see Figure 2).

$$\begin{pmatrix}
\alpha & \beta & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \alpha & \beta & \cdot & \cdot & \cdot \\
\gamma & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot \\
\bullet & \gamma & \bullet & \bullet & \cdot & \cdot & \cdot \\
\bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot \\
\bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}$$

Fig. 2.

If $|C \setminus A| = 8$, then all eight positions in both sets S_α, S_β are occupied by colours of $C \setminus A$. Further, all ten bullet positions in Figure 2, which are positions of vertices in $(N(V_\alpha) \cup N(V_\beta)) \cap N(V_\gamma)$, are occupied by colours of $(C \setminus A) \setminus \{\gamma\})$, and so $\operatorname{exc}(\gamma) \geq 10 - (8-1) = 3$, a contradiction.

Suppose that $|C \setminus A| = 7$ (and $A| = 12$). For $m \in \{2, 3+, 4+\}$ and $n \in [0,2]$ let C_m^n be the set of colours in C_m having n copies in $[5,6] \times [5,7]$, and let $c_m^n = |C_m^n|$. If $\delta \in C_2^1 \cup C_3^2$, then, since the pairs $\{\delta, \alpha\}$, $\{\delta, \beta\}$ and $\{\delta, \gamma\}$ are good, δ must appear in $\{2\} \times [1,2]$, and so $c_2^1 + c_3^2 \leq 2$; further, $c_2^2 = 0$. Using Claim 3.2.9 we obtain

$$\begin{aligned}
6 &= c_2^1 + c_3^1 + c_{4+}^1 + 2c_3^2 + 2c_{4+}^2 \leq c_2^1 + c_3^2 + \sum_{n=0}^{2} (c_3^n + c_{4+}^n) + c_{4+}^2 \\
&\leq c_2^1 + c_3^2 + c_{3+} + c_{4+} \leq 2 + 4 = 6,
\end{aligned} \tag{3.1}$$

which implies

$$c_3^0 = c_{4+}^0 = c_{4+}^1 = 0, \tag{3.2}$$

$$c_{4+} = c_{4+}^2, \tag{3.3}$$

$$c_2^1 + c_3^2 = 2, \tag{3.4}$$

$c_{3+} + c_{4+} = 4$, and so, by Claim 3.2.10, $c_{5+} = 0$.

For $\delta \in \{\alpha, \beta\}$ choose a set $S_{\delta'} \subseteq S_\delta$ with $|S_{\delta'}| = 7$ occupied by seven distinct colours of $C \setminus A$, and let

$$P = ([3,6] \times [1,4]) \setminus (S_{\alpha'} \cup S_{\beta'});$$

then $|P| = 2$. Since

$$|N(V_\gamma) \cap ([3,6] \times [1,4])| = 10,$$

we have

$$2 = \mathrm{exc}(\gamma) \geq 10 - (|P| + |(C \setminus A) \setminus \{\gamma\}| = 4 - |P| = 2,$$

hence both positions in P are necessarily occupied by a colour of A, and all sets $S_{\alpha'}, S_{\beta'}, P$ are unique. We express this property of γ by saying that γ is A-*exact*. Besides that, the two positions in P are occupied by two distinct colours of A, say λ and μ; indeed, otherwise the colour of A, which occupies both positions in P, by (3.3), would be a 5+colour, a contradiction. Let $P = \{(i_\lambda, j_\lambda), (i_\mu, j_\mu)\}$, where $\lambda = (M)_{i_\lambda, j_\lambda}$ and $\mu = (M)_{i_\mu, j_\mu}$. The excess of both α, β is 2, therefore $(j_\lambda, j_\mu) \in \{1,3\} \times \{2,4\}$ (a colour occupying a position in P contributes to the excess of either α or β, and α, β are contributing to the excess of each other).

The above reasoning concerning γ can be repeated to prove that any colour in $(C \setminus A) \cap C_2$ is A-exact.

Suppose that $\varepsilon \in (C \setminus A) \cap C_2$; then ε is A-exact and $\varepsilon \in \mathbb{C}_2(j', l')$, where $(j', l') \in \{1,3\} \times \{2,4\}$. Let $\{l_\lambda, l_\mu\} = [1,4] \setminus \{j_\lambda, j_\mu\}$.

Assume first that $i_\lambda = i_\mu$. By Claim 3.4,

$$|(C \setminus A) \cap \mathbb{C}_2(j_\lambda, j_\mu)| \leq 2.$$

If $(j', l') \neq (j_\lambda, j_\mu)$, then either $\varepsilon = (M)_{i_\lambda, l_\lambda}$ or $\varepsilon = (M)_{i_\lambda, l_\mu}$.

The second possibility is $i_\lambda \neq i_\mu$. By Claim 3.4,

$$|(C \setminus A) \cap \mathbb{C}_2(l_\lambda, l_\mu)| \leq 2.$$

On the other hand, if $(j', l') \neq (l_\lambda, l_\mu)$, then either $\varepsilon = (M)_{i_\lambda, j_\mu}$ or $\varepsilon = (M)_{i_\mu, j_\lambda}$.

In both cases

$$|(C \setminus A) \cap C_2| \leq 2 + 2 = 4$$

and

$$|(C \setminus A) \cap C_{3+}| \geq 7 - 4 = 3. \tag{3.5}$$

From (3.1) and (3.3) we obtain

$$(C \setminus A) \cap C_{3+} \subseteq C_3^1 \cup C_{4+}^2,$$

hence $c_3^1 + c_{4+}^2 \geq 3$. Let us show the following:

$$\text{No 3+colour occupies a position in } [3,4] \times [5,7], \text{ and } c_2^1 \geq 1. \tag{3.6}$$

Because of (3.2) and (3.3) we know that colours of $(C \setminus A) \cap C_{3+}$ appear only in $([3,6] \times [1,4]) \cup ([5,6] \times [5,7])$. If $c_{4+} \geq 1$, then, by Claim 3.2.8,

$$\begin{aligned} &3 \geq c_{3+} \geq c_3^1 + c_{4+}^2 \geq 3, \quad c_{3+} = c_3^1 + c_{4+}^2 = 3, \\ &C_{3+} = C_3^1 \cup C_{4+}^2 = (C \setminus A) \cap C_{3+}, \end{aligned}$$

$c_3^2 = 0$, $c_2^1 = 2$ (see (3.4)), and so (3.6) is true.

If $c_{4+} = 0$, then from (3.1), (3.4) and Claim 3.2.4 it follows that

$$c_3^1 + c_3^2 = 4 = c_{3+}, \quad C_{3+} = C_3^1 \cup C_3^2 = ((C \setminus A) \cap C_{3+}) \cup C_3^2$$

and, by (3.5), $c_3^1 \geq 3$; since a colour of C_3^2 is only in $(\{2\} \times [1,2]) \cup ([5,6] \times [5,7])$, $c_3^2 \leq 1$ and $c_2^1 \geq 1$, (3.6) is true again.

Now, by (3.6), six positions in $[3,4] \times [5,7]$ are occupied by six distinct 2-colours belonging to $A \setminus \{\lambda, \mu\}$, and there is a colour $\zeta \in C_2^1$, (w) $\zeta = (M)_{5,5}$, see Figure 3.

$$\begin{pmatrix} \alpha & \beta & \cdot & \cdot & \circ & \cdot & \cdot \\ \zeta/\cdot & \cdot/\zeta & \alpha & \beta & \circ & \circ & \circ \\ \gamma & \cdot & \cdot & \cdot & \cdot & \bullet & \bullet \\ \cdot & \gamma & \cdot & \cdot & \cdot & \bullet & \bullet \\ \cdot & \cdot & \cdot & \cdot & \zeta & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Fig. 3.

If a 2-colour η appears in a bullet position, then, since the pair $\{\eta, \zeta\}$ is good, the second copy of η must occupy a circle position. In such a case, however, it is easy to check that there is a set $\{\vartheta, \iota\} \subseteq A \cap C_2$ of 2-colours occupying two bullet positions and two circle positions, which contradicts either Claim 3.3 or Claim 3.4.

The case $(j, l) \neq (1, 2)$ can be treated similarly. □

Claim 3.7. *No set* $\{\alpha, \beta\} \subseteq C_2$ *is of the type* $(2^2, 2^1 1^2)$.

Proof. Let (w) $(M)_{1,1} = \alpha = (M)_{2,2}$ and $(M)_{1,2} = \beta = (M)_{2,3}$. First of all, we have $\mathbb{R}_2(1,2) = \{\alpha, \beta\}$. Indeed, if (w) $\gamma \in \mathbb{R}(1,2) \setminus \{\alpha, \beta\}$, then, by Claim 3.6, necessarily $(M)_{1,3} = \gamma = (M)_{2,1}$. Each colour $\delta \in C \setminus \mathbb{R}_2(1,2)$ occupies at least two positions in $[3,6] \times [1,3]$ (all pairs $\{\delta, \alpha\}$, $\{\delta, \beta\}$, $\{\delta, \gamma\}$ are good), hence $|C| \leq 11 + \lfloor \frac{4 \cdot 3}{2} \rfloor = 17 < 19$, a contradiction.

With $A = \mathbb{R}(1) \cup \mathbb{R}(2) \cup \mathbb{C}(2)$ any colour $\gamma \in C \setminus A$ occupies a position in $[3,6] \times \{1\}$ as well as in $[3,6] \times \{3\}$, hence $|C \setminus A| \leq 4$, $|A| \leq 16$, $|C \setminus A| = 19 - |A| \geq 3$ and $|A| \geq 15$.

Assume first that $|A| = 15$ and $|C \setminus A| = 4$, which yields $C \setminus A \subseteq C_{3+}$ (a 2-colour $\gamma \in C \setminus A$ would satisfy $\mathrm{exc}(\gamma) \geq 3$), $c_{3+} = c_3 = 4$ and $A = C_2$. For colours $\gamma = (M)_{1,3}$ and $\delta = (M)_{2,1}$ their second copies appear in

$[3,6] \times (\{2\} \cup [4,7])$, and the pair $\{\gamma,\delta\}$ is good in the corresponding 4×5 submatrix of M. However, at most one of γ,δ is in $[3,6] \times \{2\}$, hence $\{\gamma,\delta\}$ is good in the submatrix of M corresponding to $[3,6] \times [4,7]$.

If the pair $\{\gamma,\delta\}$ is column-based, then (w) $\gamma = (M)_{3,4}$ and $\delta = (M)_{4,4}$, at most one of colours in $[5,6] \times \{2\}$ belongs to $\mathbb{R}(1) \cup \mathbb{R}(2)$, hence there is a 2-colour ε and $i \in [5,6]$ such that $\varepsilon = (M)_{i,2} = (M)_{11-i,4}$ (so that both pairs $\{\varepsilon,\gamma\}$, $\{\varepsilon,\delta\}$ are good). For every colour $\zeta \notin \mathbb{C}(2) \cup \{(M)_{i,4}\}$ occupying a position in $[1,2] \times [5,7]$ (the number of such colours is at least 4) there is $\eta \in \{\gamma,\delta,\varepsilon\}$ such that the pair $\{\eta,\zeta\}$ is not good, a contradiction.

If the pair $\{\gamma,\delta\}$ is row-based, (w) $\gamma = (M)_{3,4}$ and $\delta = (M)_{3,5}$. If a colour ε occupies a position in $[4,6] \times \{2\}$ and does not belong to $\mathbb{R}(1) \cup \mathbb{R}(2)$ (there are at least two such colours), then it must appear in $\{3\} \times [6,7]$ (pairs $\{\varepsilon,\gamma\}$ and $\{\varepsilon,\delta\}$ are good), (w) $(M)_{4,2} = \varepsilon = (M)_{3,6}$ and $(M)_{5,2} = \zeta = (M)_{3,7}$. If a 2-colour η is in $\{6\} \times [4,7]$, then $\eta = (M)_{3,2}$ (all pairs $\{\eta,\vartheta\}$ with $\vartheta \in \{\gamma,\delta,\varepsilon,\zeta\}$ are good), $r_3(6) \geq 2 + 3 = 5 > c_3$, and so the colouring f_M is not proper, a contradiction.

From now on $|A| = 16$ and $|C \setminus A| = 3$. Suppose first that $C \setminus A \subseteq C_{3+}$. From $c_{3+} \leq 4$ we obtain $|A \cap C_{3+}| \leq 1$.

If $(\mathbb{R}(1) \cup \mathbb{R}(2)) \cap C_{3+} = \emptyset$, then (w) $\gamma = (M)_{3,2}$, $\delta = (M)_{4,2}$, $\varepsilon = (M)_{5,2}$ are 2-colours, and their second copies appear in $[3,6] \times [4,7]$. Let

$$J = \{j \in [4,7] : \mathbb{C}(j) \cap \{\gamma,\delta,\varepsilon\} \neq \emptyset\};$$

by Claim 3.5 we know that $2 \leq |J| \leq 3$. If

$$(i,j) \in S = \{(1,3),(2,1)\} \cup ([1,2] \times ([4,7] \setminus J)),$$

then $g(i,j,\{\gamma,\delta,\varepsilon\}) = 0$; note that $|S| = 10 - 2|J|$. On the other hand, the number of pairs $(i,j) \in [3,6] \times [4,7]$, satisfying $g(i,j,\{\gamma,\delta,\varepsilon\}) = 3$, is less than $|S|$ (at most 3 if $|J| = 3$ and at most 4 if $|J| = 2$). Thus, there is a 2-colour ζ in S and $\eta \in \{\gamma,\delta,\varepsilon\}$ such that the pair $\{\zeta,\eta\}$ is not good.

If $|(\mathbb{R}(1) \cup \mathbb{R}(2)) \cap C_{3+}| = 1$, then $c_3 = c_{3+} = 4$ and $c_2(2) = 6$.

Suppose first that both $\gamma = (M)_{1,3}$ and $\delta = (M)_{2,1}$ are 2-colours. The second copies of γ,δ are then in $[3,6] \times [4,7]$, for if not,

$$\max(\operatorname{exc}(\gamma), \operatorname{exc}(\delta)) \geq 1 + |C \setminus A| = 4.$$

If the pair $\{\gamma,\delta\}$ is column-based, then (w) $\gamma = (M)_{3,4}$ and $\delta = (M)_{4,4}$ so that $(M)_{5,2} = \varepsilon = (M)_{6,4}$ and $(M)_{6,2} = \zeta = (M)_{5,4}$ (all pairs $\{\varepsilon,\gamma\}, \{\varepsilon,\delta\}, \{\zeta,\gamma\}$, $\{\zeta,\delta\}$ are good). For $(i,j) \in [1,2] \times [5,7]$ then $g(i,j,\{\gamma,\delta,\varepsilon,\zeta\}) = 1$, and at least three positions in $[1,2] \times [5,7]$ are occupied by a 2-colour that is in $[3,6] \times [5,7]$; on the other hand, for $(i,j) \in [3,6] \times [5,7]$ we have $g(i,j,\{\gamma,\delta,\varepsilon,\zeta\}) \leq 2$, a contradiction.

If the pair $\{\gamma,\delta\}$ is row-based, then (w) $\gamma = (M)_{3,4}$ and $\delta = (M)_{3,5}$. Then $g(i,j,\{\gamma,\delta\}) = 0$ for $(i,j) \in [4,6] \times \{2\}$ and $g(i,j,\{\gamma,\delta\}) \leq 1$ for $(i,j) \in [4,6] \times [4,7]$; this leads to a contradiction, since at least one of colours in $[4,6] \times \{2\}$ has its second copy in $[4,6] \times [4,7]$.

So, one of γ, δ is a 2-colour and the other a 3-colour, (w) $\gamma \in C_2$ and $\delta \in C_3$. As above, the second copy of γ appears in $[3,6] \times [4,7]$, (w) $\gamma = (M)_{3,4}$. All colours of the set $B = \{\varepsilon, \zeta, \eta, \vartheta\}$, where $\varepsilon = (M)_{3,2}$, $\zeta = (M)_{4,2}$, $\eta = (M)_{5,2}$ and $\vartheta = (M)_{6,2}$, are 2-colours. By Claim 3.3, the second copy of a colour $\iota \in B$ does not appear in $\mathbb{C}(1) \cup \mathbb{C}(3)$, hence is in $[3,6] \times [4,7]$ and, additionally, in $\mathbb{R}(3) \cup \mathbb{C}(4)$, provided that $\iota \neq \varepsilon$ (the pair $\{\iota, \gamma\}$ is good). Then $|B \cap \mathbb{R}(3)| \leq 3$, since otherwise $\operatorname{exc}(\varepsilon) \geq 3$. So, by Claim 3.5, with $B' = \{\zeta, \eta, \vartheta\}$ we have $1 \leq |B' \cap \mathbb{C}(4)| \leq 2$.

If $|B' \cap \mathbb{C}(4)| = 2$, then (w) $\eta = (M)_{4,4}$, $\zeta = (M)_{4,5}$ (here we use Claim 3.4) and $\vartheta = (M)_{3,5}$. For both $l \in [6,7]$ then $g(2, l, B' \cup \{\gamma\}) = 0$. This leads to a contradiction, since $(M)_{2,6}, (M)_{2,7}$ are 2-colours, and $g(i, j, B' \cup \{\gamma\}) = 4$ only if $(i,j) = (6,4)$.

If $|B' \cap \mathbb{C}(4)| = 1$, then (w) $\zeta = (M)_{5,4}, \eta = (M)_{3,5}$ and $\vartheta = (M)_{3,6}$ so that $g(2, 7, B' \cup \{\gamma\}) = 0$. A contradiction follows from the fact that $g(i, j, B' \cup \{\gamma\}) \leq 3$ for each (i,j).

Now suppose that $(C \setminus A) \cap C_2 \neq \emptyset$, (w) $\gamma = (M)_{3,1} = (M)_{4,3} \in (C \setminus A) \cap C_2$. For $m \in \{2, 3, 3+\}$, $n \in [1,2]$ let C_m^n be the set of colours in C_m having n copies in $[5,6] \times [4,7]$ and $c_m^n = |C_m^n|$; then

$$c_2^1 + c_{3+}^1 + 2c_{3+}^2 = 8. \tag{3.7}$$

Since $g(i, j, \{\alpha, \beta, \gamma\}) = 0$ for $(i,j) \in [5,6] \times [4,7]$ and $g(i, j, \{\alpha, \beta, \gamma\}) = 3$ if and only if $(i,j) \in S = \{(1,3), (2,1), (3,2), (4,2)\}$, we have

$$c_2^1 + c_3^2 \leq 4. \tag{3.8}$$

Let us first show that $c_2^1 \leq 3$. Indeed, if $c_2^1 = 4$, then all pairs $\{\delta, \varepsilon\} \in \binom{C_2^1}{2}$ are good only if there is $i \in [5,6]$ such that $C_2^1 \subseteq \mathbb{R}(i)$. This immediately implies $c_{3+}^2 = 0$ and, by Claim 3.2.4 and (3.7), $4 \geq c_{3+} \geq c_{3+}^1 = 4$, $c_{3+} = 4$ and $C_{3+} \subseteq \mathbb{R}(11 - i)$. Then $\delta = (M)_{11-i,2} \in C_2$, the second copy of δ is in $[3,4] \times [4,7]$ (by Claim 3.3), hence at least one of pairs $\{\delta, \varepsilon\}$ with $\varepsilon \in C_2^1$ is not good, a contradiction.

Further, we prove that

$$c_{3+}^1 + c_{3+}^2 = c_{3+}, \tag{3.9}$$

which is equivalent to

$$C_{3+}^1 \cup C_{3+}^2 = C_{3+}. \tag{3.10}$$

If $c_{4+} \geq 1$, then Claim 3.2.8 yields $c_{3+} \leq 3$. Because of (3.7) we obtain

$$\begin{aligned} 2(c_{3+}^1 + c_{3+}^2) &= 8 + c_{3+}^1 - c_2^1, \\ c_{3+}^1 + c_{3+}^2 &= \frac{1}{2}(8 + c_{3+}^1 - c_2^1) \geq \frac{1}{2}(8 - 3) = \frac{5}{2}, \\ 3 \geq c_{3+} &\geq c_{3+}^1 + c_{3+}^2 \geq 3, \end{aligned}$$

and so (3.9) is true under the assumption $c_{4+} \geq 1$ (implying $c_{3+} = 3$).

On the other hand, $c_{4+} = 0$ implies $c_{3+} = c_3 = 4$ (Claim 3.2.7). In this case, using (3.7) and (3.8), we see that

$$8 = (c_2^1 + c_{3+}^2) + (c_{3+}^1 + c_{3+}^2) \leq 4 + c_{3+} = 8,$$

hence

$$c_{3+} = 4 = c_2^1 + c_{3+}^2 = c_{3+}^1 + c_{3+}^2,$$

and (3.9) holds again.

Note that now necessarily

$$|(C \setminus A) \cap C_2| = 1.$$

Indeed, $|(C \setminus A) \cap C_2| = 3$ is impossible by Claim 3.5, since in such a case

$$c_2(1,3) \geq |(C \setminus A) \cap C_2| = 3.$$

Moreover, by Claims 3.3 and 3.4, the assumption $|(C \setminus A) \cap C_2| = 2$ would mean that for the unique colour $\delta \in (C \setminus A) \cap C_{3+}$ there is $i \in [5,6]$ such that $(M)_{i,1} = \delta = (M)_{11-i,3}$; however, according to (3.10), δ has an exemplar in $[5,6] \times [4,7]$, and so the colouring f_M is not proper, a contradiction.

Thus

$$|(C \setminus A) \cap C_{3+}| = 2. \tag{3.11}$$

Because of a reasoning analogous to that above we see that each colour of $(C \setminus A) \cap C_{3+}$ occupies exactly one position in $[5,6] \times \{1,3\}$,

$$(C \setminus A) \cap C_{3+} = C_{3+}^1, \tag{3.12}$$

and then, using (3.10),

$$C_{3+} \subseteq \mathbb{R}(5) \cap \mathbb{R}(6). \tag{3.13}$$

Now if $c_{4+} \geq 1$ (and, consequently, $c_{3+} = 3$, which we have seen already), then, by (3.9), (3.11) and (3.12), $c_{3+}^1 = 2$ and $c_{3+}^2 = 1$; since $c_2^1 \leq 3$, in such a case $c_2^1 + c_{3+}^1 + 2c_{3+}^2 \leq 7$ in contradiction with (3.7).

Therefore, in the rest of the proof of Claim 3.7 we work with $c_{4+} = 0$, $c_{3+} = 4$, $c_{3+}^1 = 2$, $c_3^2 = c_{3+}^2 = 2$ and $c_2^1 = 2$, see (3.7), (3.9), (3.11), (3.12). Moreover, all positions in S are occupied by colours of $C_2^1 \cup C_3^2$. If $\delta = (M)_{i,j} \in C_2^1$ with $(i,j) \in \{(1,3),(2,1)\}$, then, because of (3.11), (3.12) and (3.13),

$$\operatorname{exc}(\delta) \geq 1 + |(C \setminus A) \cap C_{3+}| = 1 + c_{3+}^1 = 3,$$

a contradiction.

Thus $\{(M)_{1,3}, (M)_{2,1}\} \subseteq C_{3+}^2$, and for a 2-colour ε occupying a position in $[5,6] \times \{1,3\}$ (there are two such colours), by (3.13) we have $\operatorname{exc}(\varepsilon) \geq c_{3+} - 1 = 3$, a contradiction again. $\square$

Claim 3.8. *If* $i,k \in [1,6]$*,* $i \neq k$*, then* $r_2(i,k) \leq 2$*.*

Proof. The assumption $r_2(i,k) \geq 3$ would be in contradiction with Claim 3.6 or Claim 3.7. $\square$

Claim 3.9. *No set* $\{\alpha,\beta,\gamma\}\subseteq C_2$ *is of the type* $(3^1 2^1 1^1, 3^1 2^1 1^1)$.

Proof. Having in mind Claim 3.4 (or else Claim 3.7), assume (w) $(M)_{1,1}=\alpha=(M)_{2,2}$, $(M)_{1,2}=\beta=(M)_{2,1}$ and $(M)_{1,3}=\gamma=(M)_{3,1}$. Let A^1 be the set of colours occupying a position in

$$(\{1\}\times[4,7])\cup([4,6]\times\{1\})\cup\{(2,3),(3,2)\},$$

A^n the set of colours in

$$(\{n\}\times[4,7])\cup([4,6]\times\{n\})\quad\text{for } n=2,3,$$

$A^n_m=C_m\cap A^n$ and $a^n_m=|A^n_m|$ for $m\in\{2,3+\}$, $n\in[1,3]$. Then

$$|A^1|=a^1_2+a^1_{3+}=9, \tag{3.14}$$

$$|A^2|=a^2_2+a^2_{3+}=7, \tag{3.15}$$

$$A_1\cap A_2=\emptyset, \tag{3.16}$$

since otherwise $\operatorname{exc}(\alpha)\geq 3$. Moreover, $|A^3|\leq 7$ and $A^2\subseteq A^3$ (each pair $\{\gamma,\eta\}$ with $\eta\in A_2$ is good), hence, by (3.15),

$$A_2=A_3. \tag{3.17}$$

Let us show that distinct colours $\delta=(M)_{2,3}$, $\varepsilon=(M)_{3,2}$ (a consequence of (3.14)) satisfy

$$\{\delta,\varepsilon\}\subseteq C_{3+}. \tag{3.18}$$

Indeed, if $\eta=(M)_{i,5-i}\in\{\delta,\varepsilon\}\cap C_2$ for some $i\in[2,3]$ and (w) $\eta=(M)_{4,4}$, then all colours appearing in

$$(\{i\}\times[4,7])\cup([4,6]\times\{5-i\})\cup\{(5-i,4),(4,i)\}$$

belong to $A^2\setminus\{\eta\}$, hence, by (3.15) and (3.17),

$$\operatorname{exc}(\eta)\geq 9-(7-1)=3,$$

a contradiction.

Further, with $\zeta=(M)_{3,3}$ we have

$$\zeta\in C_{3+}\cap A^1. \tag{3.19}$$

To see it realise first that, since the pair $\{\zeta,\alpha\}$ is good and f_M is proper, we get $\zeta\notin A^2\cup\{\delta,\varepsilon\}$ and $\zeta\in A^1$. Moreover, $\zeta\in C_{3+}$, because the assumption $\zeta\in\mathbb{R}_2(1)$ ($\zeta\in\mathbb{C}_2(1)$) contradicts Claim 3.7 (Claim 3.4, respectively).

By (3.14), (3.15) and Claim 3.2.4, we have

$$a^1_2+a^2_2\geq(9+7)-c_{3+}\geq 12. \tag{3.20}$$

Further, by (3.14), (3.15) and (3.18)–(3.20),

$$5\leq a^1_2\leq 6, \tag{3.21}$$

$$6\leq a^2_2\leq 7. \tag{3.22}$$

Consider a colour $\eta \in A^1_2 \cap \mathbb{R}(1)$ (from(3.21) we see that there are at least two such colours), (w) $\eta = (M)_{1,j} = (M)_{4,l}$. Then from

$$g(1, j, A^2_2) \le g(1, j, A^2) = 2 \quad \text{and} \quad g(4, l, A^2_2) \le g(4, l, A^2) \le 4$$

it follows that $a^2_2 \le 2 + 4 = 6$, hence, by (3.15), (3.21), (3.22), $a^1_2 = a^2_2 = 6$ and $a^2_{3+} = 1$.

Suppose first $\zeta \in \mathbb{C}(1)$ so that all positions in $\{1\} \times [4, 7]$ are occupied by colours of A^1_2. If $\eta = (M)_{1,j}$, $j \in [4, 7]$, then proceeding similarly as above we find that both positions in $[2, 3] \times \{j\}$ are occupied by colours of A^2_2. Thus $\mathbb{R}_2(2, 3)$ consists of at least two colours of A^2_2. By Claim 3.8 we obtain $r_2(2, 3) = 2$, (w) $(M)_{2,4} = \vartheta = (M)_{3,5}$ and $(M)_{2,5} = \iota = (M)_{3,4}$. Now $\kappa = (M)_{1,6}$ satisfies $\kappa \in \mathbb{C}(4) \cup \mathbb{C}(5)$ (the pair $\{\kappa, \vartheta\}$ is good) and, analogously, $\lambda = (M)_{1,7} \in \mathbb{C}(4) \cup \mathbb{C}(5)$. By Claim 3.6, the copies of κ, λ that are in $[4, 6] \times [4, 5]$ do not share a row, (w) one of them is in $\mathbb{R}(4)$ and the other in $\mathbb{R}(5)$. Then, however, reasoning similarly as above again, all positions in $[4, 5] \times [2, 3]$ are occupied by colours of A^2_2. Consequently, the unique colour of A^2_{3+} is $(M)_{6,2} = (M)_{6,3}$, and f_M is not proper.

If $\zeta \in \mathbb{R}(1)$, (w) $\zeta = (M)_{1,7}$, then all positions in

$$(\{1\} \times [4, 6]) \cup ([4, 6] \times \{1\})$$

are occupied by colours of A^1_2, which implies that all positions in

$$([2, 3] \times [4, 6]) \cup ([4, 6] \times [2, 3])$$

are occupied by colours of A^2_2. So, the unique colour of A^2_{3+} is $(M)_{2,7} = (M)_{3,7}$, a contradiction. ☐

4. FINAL ANALYSIS

We are now ready to do the final analysis for proving Theorem 1.3. Suppose (w) that $r_2(1) \ge r_2(i)$ for $i \in [2, 6]$, which, by Claim 3.2.3, implies

$$7 \ge r_2(1) \ge \left\lceil \frac{2c_2}{6} \right\rceil \ge \left\lceil \frac{30}{6} \right\rceil = 5. \tag{4.1}$$

Given $r_2(1)$ we assume (w) that the sequence $S = (r_2(1, i))^6_{i=2}$ is nonincreasing. Since $r_2(1) \in [5, 7]$, we have $r_2(1, 2) \ge \lceil \frac{r_2(1)}{5} \rceil \ge 1$, $r_2(1, 6) \le \lfloor \frac{r_2(1)}{5} \rfloor = 1$, and Claim 3.8 yields $r_2(1, 2) \le 2$. We suppose (w) that

$$j \in [1, r_2(1)] \Rightarrow (M)_{1,j} \in C_2,$$

and, more precisely,

$$\begin{aligned} &(M)_{1,1} = \alpha,\ (M)_{1,2} = \beta,\ (M)_{1,3} = \gamma,\ (M)_{1,4} = \delta,\ (M)_{1,5} = \varepsilon, \\ &r_2(1) \ge 6 \Rightarrow (M)_{1,6} = \zeta, \quad r_2(1) = 7 \Rightarrow (M)_{1,7} = \eta. \end{aligned}$$

Let p be the smallest integer in $[2,6]$ such that $r_2(1,i) \leq 1$ for every $i \in [p,6]$; p is correctly defined since $r_2(1,6) \leq 1$. Then

$$r_2(1,i) = 1 \Leftrightarrow i \in [p, r_2(1) + 3 - p],$$

and, counting the number of positions in $[2,6] \times [1,7]$ occupied by colours of $\mathbb{R}_2(1)$, we obtain

$$2(p-2) \leq r_2(1) \leq 2(p-2) + (7-p),$$

which yields

$$r_2(1) - 3 \leq p \leq \left\lfloor \frac{r_2(1)+4}{2} \right\rfloor \leq 5. \tag{4.2}$$

Moreover, because of Claims 3.6 and 3.7 we have

$$\begin{aligned} p \geq 3 &\Rightarrow ((M)_{2,1} = \beta \wedge (M)_{2,2} = \alpha), \\ p \geq 4 &\Rightarrow ((M)_{3,3} = \delta \wedge (M)_{3,4} = \gamma), \\ p = 5 &\Rightarrow ((M)_{4,5} = \zeta \wedge (M)_{4,6} = \varepsilon). \end{aligned}$$

Let $q_j = |\mathbb{R}_2(1) \cap \mathbb{C}(j)|$ for $j \in [1,7]$. By Claim 3.9 we know that a 2-colour μ, which occupies a position in $\{1\} \times [2p-3, r_2(1)]$, satisfies $\mu \notin \mathbb{C}(j)$ for every $j \in [1, 2p-4]$, hence $q_j = 2$ for any $j \in [1, 2p-4]$, and

$$\sum_{j=2p-3}^{7} q_j = 2[r_2(1) - (2p-4)] = 2r_2(1) + 8 - 4p; \tag{4.3}$$

further,

$$j \in [2p-3, 7] \Rightarrow q_j \leq 3, \tag{4.4}$$

since with $q_j \geq 4$ and $\mu \in \mathbb{R}_2(1) \cap \mathbb{C}(j)$ for some $j \in [2p-3,7]$ we have $\operatorname{exc}(\mu) \geq q_j - 1 \geq 3$. Notice also that

$$j \in [2p-3, r_2(1)] \Rightarrow 1 \leq q_j \leq \min(3, r_2(1) + 4 - 2p), \tag{4.5}$$

because 2-colours occupying a position in $[1,6] \times \{j\}$ are distinct from $2p-4$ (2-)colours appearing in $\{1\} \times [1, 2p-4]$. Moreover, we assume (w) that the sequence $(q_j)_{j=2p-3}^{r_2(1)}$ is nonincreasing, and that, if $(r_2(1), p) = (5,2)$, the sequence $(q_1, q_2, q_3, q_4, q_5)$ is nonincreasing.

For every pair $(r_2(1), p)$ obeying (4.1) and (4.2), we analyse the set $\mathcal{Q}(r_2(1), p)$ of sequences $(q_j)_{j=2p-3}^{7}$ satisfying all restrictions (4.3)–(4.5). More precisely, we show that the assumption that M is characterised by an arbitrary sequence $Q \in \mathcal{Q}(r_2(1), p)$ leads to a contradiction, mostly because of $\Sigma \geq 13$ (a contradiction to Claim 3.2.10) or the existence of a line of M containing at least five copies of 3+colours (by Claim 3.2.4 then the colouring f_M is not proper).

The structure of the sets $\mathcal{Q}(r_2(1),p)$ with $(r_2(1),p) \neq (5,2)$ follows:

$$\begin{aligned}
\mathcal{Q}(7,5) &= \emptyset,\\
\mathcal{Q}(7,4) &= \{(3,2,1),(2,2,2)\},\\
\mathcal{Q}(6,5) &= \{(0)\},\\
\mathcal{Q}(6,4) &= \{(2,2,0),(2,1,1),(1,1,2)\},\\
\mathcal{Q}(6,3) &= \{(3,3,1,1,0),(3,2,2,1,0),(3,2,1,1,1),(3,1,1,1,2),(2,2,2,2,0),\\
&\qquad (2,2,2,1,1),(2,2,1,1,2),(2,1,1,1,3)\},\\
\mathcal{Q}(5,4) &= \{(1,1,0)\},\\
\mathcal{Q}(5,3) &= \{(3,2,1,0,0),(3,1,1,1,0),(2,2,2,0,0),(2,2,1,1,0),(2,1,1,2,0),\\
&\qquad (2,1,1,1,1),(1,1,1,3,0),(1,1,1,2,1)\}.
\end{aligned}$$

As we shall see later, it is not necessary to know the structure of $\mathcal{Q}(5,2)$ explicitly.

Our analysis is organised according to the following rules: All *visible* colours in M (those represented by Greek alphabet letters) are 2-colours, and both copies of a visible colour are present in M. Asterisk entries in M represent 3+colours. Some asterisk entries appear in M by definition, *e.g.*, each asterisk entry in the first row of M occupies a position in $\{1\} \times [r_2(1)+1,7]$. Another reason why an asterisk entry appears in M is that, if the corresponding position is occupied by a 2-colour λ, then putting another copy of λ to a *free* position (i.e., one that is not occupied by a visible colour) in any proper way (so that the resulting partial vertex colouring f' of $K_6 \,\square\, K_7$ is proper) leads to a situation, in which no continuation of f' to a proper complete vertex colouring of $K_6 \,\square\, K_7$ is possible, because at least one pair $\{\lambda,\mu\}$, where μ is a visible colour, is not good.

To simplify the description of matrices appearing in our analysis we frequently use the notation "$Q = \tilde{Q}$, Figure xy:" or "$Q = \tilde{Q}$, (w) Figure xy:", where $\tilde{Q} \in \mathcal{Q}((r_2(1),p))$. It means that the situation, in which M is characterised by the sequence $\tilde{Q}$, is analysed in Figure xy (and possibly Proposition 2.2 is involved).

If $Q = (3,2,1)$, then (w) $(M)_{4,5} = \zeta$, $(M)_{5,5} = \eta$ and $(M)_{6,6} = \varepsilon$, hence the set $\{\varepsilon,\zeta\}$ is of the type $(2^1 1^2, 2^2)$, which contradicts Claim 3.4.

In the case $Q = (2,2,2)$ we are (w) in the situation of Figure 4. If a 2-colour μ occupies a position in $\{k\} \times [2l-1,2l]$ for some $k \in [4,6]$ and $l \in [1,2]$, then $\mu = (M)_{4-l,h(k)}$, where $h(k) = \frac{1}{2}(3k^2 - 31k + 90)$, and $\nu \in C_{3+}$ for each colour ν occupying a position in $([4,6] \setminus \{k\}) \times [5-2l, 6-2l]$ (with $\nu \in C_2$ the pair $\{\mu,\nu\}$ is not good). As a consequence, at least nine positions in $[4,6] \times [1,4]$ are occupied by 3+colours. Besides that, if $\mu = (M)_{i,j} \in C_2$ with $(i,j) \in \{(2,3),(2,4),(3,1),(3,2)\}$, the second copy of μ must occupy one of the positions $(4,7),(5,5),(6,6)$. Altogether we have

$$\Sigma \geq 3 + 9 + 1 = 13.$$

$Q = (0)$, Figure 5: Because of

$$c_2(7) \geq 6 - c_{3+} \geq 2$$

we suppose (w) $\eta = (M)_{2,7} \in C_2$ so that η is in $\{(3,5),(3,6),(4,3),(4,4)\}$.

Under the assumption $\eta \in \mathbb{R}(3)$ we have (w) $\eta = (3,5)$. Let C_2' be the set of 2-colours occupying a position in $[5,6] \times [1,6]$; the inequality $c_{3+} \leq 4$ implies $|C_2'| \geq 6$. If $\mu \in C_2'$, then from the fact that each pair $\{\mu, \nu\}$ with $\nu \in C_2'' = \{\alpha, \beta, \gamma, \delta, \eta\}$ is good one easily gets that the second copy of μ occupies a position in $[2,3] \times [1,6]$. As $g(4,7,C_2'') = 0$ and $g(i,j,C_2'') \leq 5$ provided that $(i,j) \in [2,3] \times [1,6]$ is a dot position, we obtain $\omega = (M)_{4,7} \in C_{3+}$ (notice that $\omega \notin C_2'$), hence $(M)_{3,7} \in C_2$. Then $\operatorname{exc}(\eta) \geq 4$, since we can uncolour vertices (i,j) with $i \in [2,3]$ and $(M)_{i,j} \in C_{3+}$ (here we use $r_{3+}(i) \geq 1$ and $C_{3+} \subseteq \mathbb{C}(7)$), as well as the vertices $(5,5), (6,5)$ (independently from the frequencies of $(M)_{5,5}$ and $(M)_{6,5}$) without affecting the completeness of the colour class η in the resulting partial colouring.

In the case $\eta \in \mathbb{R}(4)$ we obtain a contradiction similarly as above.

The assumption $Q = (2,2,0)$ means that (w) $(M)_{4,5} = \zeta$ and $(M)_{5,6} = \varepsilon$, hence the type of the set $\{\varepsilon, \zeta\}$ is $(2^1 1^2, 2^2)$ in contradiction to Claim 3.4.

For $Q = (2,1,1)$ the situation is (w) depicted on Figure 6. If $\lambda = (M)_{6,j} \in C_2$, where $j \in [2k-1, 2k]$ with $k \in [1,2]$, then $\lambda = (M)_{4-k,5}$. As a consequence, $r_{3+}(6) = 4$, $\eta = (M)_{6,5} \in C_2$, and with $\mu = (M)_{2,5}$, $\nu = (M)_{3,5}$ we have $\{\mu, \nu\} \subseteq \mathbb{R}(6)$. Then, however, $\operatorname{exc}(\eta) \geq 3$ (μ, ν and at least one 3+colour contribute to the excess of η).

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & \zeta & \eta \\ \beta & \alpha & . & . & . & . & . \\ . & . & \delta & \gamma & . & . & . \\ . & . & . & . & \zeta & * & . \\ . & . & . & . & . & \eta & * \\ . & . & . & . & * & . & \varepsilon \end{pmatrix}$$

Fig. 4.

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & \zeta & * \\ \beta & \alpha & . & . & . & . & . \\ . & . & \delta & \gamma & . & . & . \\ . & . & . & . & \zeta & \varepsilon & . \\ . & . & . & . & . & . & * \\ . & . & . & . & . & . & * \end{pmatrix}$$

Fig. 5.

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & \zeta & * \\ \beta & \alpha & . & . & . & . & . \\ . & . & \delta & \gamma & . & . & . \\ . & . & . & . & \zeta & * & . \\ . & . & . & . & . & . & \varepsilon \\ . & . & . & . & . & * & * \end{pmatrix}$$

Fig. 6.

$Q = (1,1,2)$, Figure 7: Similarly as above we see that $r_{3+}(6) = 4$, $\eta = (M)_{6,7} \in C_2$, $\{(M)_{2,7}, (M)_{3,7}\} \subseteq \mathbb{R}_2(6)$, and so $\operatorname{exc}(\eta) \geq 4$.

If $Q = (3,3,1,1,0)$, then (w) $(M)_{3,3} = \delta, (M)_{4,3} = \varepsilon$ and $\gamma \in \{(M)_{5,4}, (M)_{6,4}\}$ so that the type of the set $\{\gamma, \delta\}$ contradicts Claim 3.4.

If $Q = (3,2,2,1,0)$, then, having in mind Claim 3.4, we are (w) in the situation of Figure 8. Further, $\eta = (M)_{2,7} \in C_2$ and $\vartheta = (M)_{5,7} \in C_2$, which implies $\eta = (M)_{5,3}$ and $\vartheta = (M)_{2,3}$. Consequently, both positions in $\{(3,4),(4,6)\}$ are occupied by 3+colours, and, provided that $\mu = (M)_{i,j} \in C_2$ for some $(i,j) \in [3,6] \times [1,2]$, then $(i,j) \in \{(5,1),(5,2)\}$ and $\mu = (M)_{6,3}$. Therefore,

$$\Sigma \geq 4 + 2 + 7 + r_{3+}(2) \geq 14.$$

$Q = (3,2,1,1,1)$, (w) Figure 9 (using Claim 3.4 again): If a bullet position is occupied by a 2-colour μ, then the second copy of μ occupies a dot position. Therefore,

$$\Sigma \geq 4 + (19 - 6) = 17.$$

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & \zeta & * \\ \beta & \alpha & . & . & . & . & . \\ . & . & \delta & \gamma & . & . & . \\ . & . & . & . & * & . & \varepsilon \\ . & . & . & . & . & * & \zeta \\ . & . & . & . & * & * & . \end{pmatrix}$$

Fig. 7.

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & \zeta & * \\ \beta & \alpha & . & . & . & . & . \\ . & . & \delta & . & . & . & * \\ . & . & \zeta & . & . & . & * \\ . & . & . & \varepsilon & . & . & . \\ . & . & . & . & \gamma & . & * \end{pmatrix}$$

Fig. 8.

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & \zeta & * \\ \beta & \alpha & . & . & \bullet & \bullet & \bullet \\ \bullet & \bullet & \delta & . & * & \bullet & * \\ \bullet & \bullet & \varepsilon & . & * & \bullet & \bullet \\ \bullet & \bullet & . & \zeta & \bullet & \bullet & \bullet \\ \bullet & \bullet & . & . & \bullet & \bullet & \gamma \end{pmatrix}$$

Fig. 9.

$Q = (3, 1, 1, 1, 2)$, (w) Figure 10: Analogously as in the case of Figure 9 we obtain

$$\Sigma \geq 5 + (19 - 6) = 18.$$

Under the assumption $Q = (2, 2, 2, 2, 0)$ we have $g(i, 7, \{\alpha, \beta, \gamma, \delta, \varepsilon, \zeta\}) = 1$ for any $i \in [3, 6]$ and $g(k, l, \{\alpha, \beta, \gamma, \delta, \varepsilon, \zeta\}) \leq 4$ for any position $(k, l) \in [2, 6] \times [1, 6]$ occupied by a colour of $C \setminus \{\alpha, \beta, \gamma, \delta, \varepsilon, \zeta\}$, hence $c_{3+}(7) \geq 5$, a contradiction.

If $Q = (2, 2, 2, 1, 1)$, then because of Claim 3.4 (w) there are two possibilities for the structure of M, see Figures 11 and 12.

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & \zeta & * \\ \beta & \alpha & . & \bullet & \bullet & \bullet & . \\ \bullet & \bullet & \delta & * & \bullet & \bullet & . \\ \bullet & \bullet & \varepsilon & \bullet & * & \bullet & . \\ \bullet & \bullet & . & * & * & \bullet & \gamma \\ \bullet & \bullet & . & \bullet & \bullet & \bullet & \zeta \end{pmatrix}$$

Fig. 10.

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & \zeta & * \\ \beta & \alpha & . & . & . & . & . \\ . & . & \delta & \bullet & . & \bullet & \bullet \\ . & . & . & \varepsilon & \bullet & \bullet & \bullet \\ . & . & \bullet & . & \gamma & \bullet & \bullet \\ . & . & . & . & . & * & \zeta \end{pmatrix}$$

Fig. 11.

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & \zeta & * \\ \beta & \alpha & . & . & . & . & . \\ . & . & \delta & * & . & * & . \\ . & . & . & \varepsilon & . & . & * \\ . & . & . & . & \zeta & * & . \\ . & . & . & . & . & . & \gamma \end{pmatrix}$$

Fig. 12.

In the case of Figure 11 a bullet position can be occupied by a 2-colour only if the second copy of that colour appears in $\{2\} \times [3, 5]$. A position in $\{(2, 6), (2, 7), (6, 1), (6, 2)\}$ is occupied by a 2-colour only if the second copy of that colour is in $\{(3, 5), (4, 3), (5, 4)\}$. Further, at most one of the two colours in $\{i\} \times [1, 2]$ with $i \in [3, 5]$ is a 2-colour (which is in $\{6\} \times [3, 5]$). Therefore

$$\Sigma \geq 2 + (9 - 3) + (4 - 3) + 3 \cdot 1 + r_{3+}(2) \geq 13.$$

In the situation of Figure 12 let

$$k = \max(i \in \{2, 3, 5\} : (M)_{i,7} \in C_2) \quad \text{and} \quad \eta = (M)_{k,7}.$$

The assumption $k = 2$ implies $\eta \in \{(M)_{3,5}, (M)_{5,4}\}$.

If $\eta = (M)_{3,5}$, see Figure 13, then

$$\Sigma \geq 13 + r_{3+}(2) \geq 14.$$

In the case $\eta = (M)_{5,4}$ depicted in Figure 14 we have $r_{3+}(3) \geq 5$.

If $k = 3$ (Figure 15), then $\eta = (M)_{2,5}$, and from $r_{3+}(2) \geq 1$ it follows that $\text{exc}(\alpha) \geq 3$, a contradiction.

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & \zeta & * \\ \beta & \alpha & . & . & . & . & \eta \\ . & . & \delta & * & \eta & * & * \\ * & * & . & \varepsilon & . & . & * \\ * & * & . & . & \zeta & * & * \\ * & * & . & . & . & . & \gamma \end{pmatrix}$$

Fig. 13.

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & \zeta & * \\ \beta & \alpha & . & . & . & . & \eta \\ * & * & \delta & * & . & * & * \\ . & . & . & \varepsilon & . & . & * \\ . & . & . & \eta & \zeta & * & * \\ * & * & . & . & . & . & \gamma \end{pmatrix}$$

Fig. 14.

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & \zeta & * \\ \beta & \alpha & . & . & \eta & . & . \\ . & . & \delta & * & . & * & \eta \\ * & * & . & \varepsilon & . & . & * \\ * & * & . & . & \zeta & * & * \\ . & . & * & . & . & . & \gamma \end{pmatrix}$$

Fig. 15.

Figure 16 corresponds to $k = 5$, requiring $\eta = (M)_{2,4}$. If $\vartheta \in C_2$ is in $\{4\} \times [1, 2]$, then $\vartheta = (M)_{5,3}$, and, if $\iota \in C_2$ is in $\{6\} \times [1, 2]$, then $\iota = (M)_{5,4}$. So, $c_{3+}(1) + c_{3+}(2) \geq 4$, which implies $\mathrm{exc}(\alpha) \geq 3$.

In the case $Q = (2, 2, 1, 1, 2)$, using Claim 3.4, (w) the description by Figure 17 applies. Claim 3.9 implies that a 2-colour occupying a position in $[3, 6] \times [1, 2]$ does not appear in $\{2\} \times [3, 7]$. Therefore, for any $i \in [3, 6]$ at most one of the positions in $\{i\} \times [1, 2]$ is occupied by a 2-colour; as a consequence of $c_{3+} \leq 4$ and $r_{3+}(2) \geq 1$ then $\mathrm{exc}(\alpha) \geq 3$.

If $Q = (2, 1, 1, 1, 3)$, then (w) we have the situation of Figure 18 with

$$\Sigma \geq 5 + (19 - 6) = 18$$

(reasoning as in Figure 9).

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & \zeta & * \\ \beta & \alpha & . & \eta & . & . & . \\ * & * & \delta & * & . & * & . \\ . & . & . & \varepsilon & . & . & * \\ . & . & . & . & \zeta & * & \eta \\ . & . & . & . & . & * & \gamma \end{pmatrix}$$

Fig. 16.

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & \zeta & * \\ \beta & \alpha & . & . & . & . & . \\ . & . & \delta & . & . & * & . \\ . & . & . & \varepsilon & * & . & . \\ . & . & * & . & * & . & \gamma \\ . & . & . & . & . & * & \zeta \end{pmatrix}$$

Fig. 17.

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & \zeta & * \\ \beta & \alpha & . & \bullet & \bullet & \bullet & . \\ \bullet & \bullet & \delta & \bullet & \bullet & \bullet & . \\ \bullet & \bullet & . & \bullet & * & * & \gamma \\ \bullet & \bullet & . & \bullet & * & \bullet & \varepsilon \\ \bullet & \bullet & . & \bullet & \bullet & * & \zeta \end{pmatrix}$$

Fig. 18.

The assumption $r_2(1) = 5$ implies $r_2(i) = 5$ and $r_{3+}(i) = 2$ for each $i \in [1, 6]$, hence

$$c_2 = \frac{1}{2} \sum_{i=1}^{6} r_2(i) = 15,$$

and, by Claims 3.2.6, 3.2.7, $c_{3+} = c_3 = 4$.

If $Q = (1, 1, 0)$, then we are (w) in the situation of Figure 19. Each colour of $\mathbb{C}_2(7)$ has its second copy in $[2, 4] \times [1, 6]$, hence at least

$$5 + 2c_2(7) + \sum_{i=2}^{4} r_{3+}(i) = 11 + 2c_2(7) \geq 15$$

positions in $[2, 4] \times [1, 7]$ are occupied by colours of $\{\alpha, \beta, \gamma, \delta, \varepsilon\} \cup \mathbb{C}_2(7) \cup C_{3+}$. Since a colour in $\mathbb{R}_2(5) \cup \mathbb{R}_2(6)$ has its second copy in $[2, 4] \times [1, 6]$, we have

$$r_2(5) + r_2(6) \leq 18 - (15 - 3) = 6$$

and

$$4 = r_{3+}(5) + r_{3+}(6) = 14 - [r_2(5) + r_2(6)] \geq 8,$$

a contradiction.

If $Q = (3,2,1,0,0)$, then the set $\{\gamma, \delta\} \subseteq C_2$ is of the type $(2^1 1^2, 2^2)$, which contradicts Claim 3.4.

$Q = (3,1,1,1,0)$, (w) Figure 20: A bullet position can be occupied by a colour $\mu \in C_2$ only if $\mu = (M)_{2,3}$. That is why $r_{3+}(6) \geq 3$, a contradiction.

$Q = (2,2,2,0,0)$, (w) Figure 21: If a bullet position is occupied by a colour $\mu \in C_2$, then $\mu \in \{(M)_{3,5}, (M)_{4,3}, (M)_{5,4}\}$. One can easily see that if $i \in [3,5]$, then at most one of colours in $\{i\} \times [6,7]$ is a 2-colour. Therefore, if $(M)_{2,j} \in C_{3+}$ for both $j = 6,7$, then

$$c_{3+}(6) + c_{3+}(7) \geq 3 \cdot 2 + 3 \cdot 1 = 9,$$

and there is $j \in [6,7]$ with $c_{3+}(j) \geq 5$, a contradiction. Thus, there is $j \in [6,7]$ with $(M)_{2,j} \in C_2$. Then, however, since $(M)_{6,1}, (M)_{6,2} \in C_2$ (a consequence of $r_{3+}(6) = 2$), the pair $\{(M)_{2,j}, (M)_{6,l}\}$ is not good for $l = 1,2$.

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & * & * \\ \beta & \alpha & . & . & . & . & . \\ . & . & \delta & \gamma & . & . & . \\ . & . & . & . & . & \varepsilon & . \\ . & . & . & . & . & . & * \\ . & . & . & . & . & . & * \end{pmatrix}$$

Fig. 19.

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & * & * \\ \beta & \alpha & . & . & . & . & . \\ . & . & \delta & \bullet & . & . & \bullet \\ . & . & \varepsilon & . & \bullet & . & \bullet \\ . & . & . & . & . & \gamma & \bullet \\ . & . & . & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

Fig. 20.

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & * & * \\ \beta & \alpha & . & . & . & \bullet & \bullet \\ . & . & \delta & . & . & . & . \\ . & . & . & \varepsilon & . & . & . \\ . & . & . & . & \gamma & . & . \\ \bullet & \bullet & . & . & . & * & * \end{pmatrix}$$

Fig. 21.

If $Q = (2,2,1,1,0)$, then (w), by Claim 3.4, the situation is depicted in Figure 22. If a 2-colour μ is in $\{(2,7),(6,1),(6,2)\}$, then $\mu \in \{(M)_{4,3}, (M)_{5,4}\}$, and if a 2-colour ν is in $\{(5,7),(6,6)\}$, then $\nu = (M)_{2,4}$. From $r_{3+}(6) = 2$ it follows that there is a 2-colour ζ in $\{6\} \times [1,2]$; as a consequence then $\omega = (M)_{2,7} \in C_{3+}$ (with $\omega \in C_2$ the pair $\{\omega, \zeta\}$ is not good), $\eta = (M)_{5,7} = (M)_{2,4} \in C_2$, $(M)_{6,6} \in C_{3+}$, and each colour, occupying a position in $\{6\} \times [1,2]$, is a 2-colour. In such a case, however, with $\vartheta = (M)_{4,3} \in \{(M)_{6,1}, (M)_{6,2}\}$ the pair $\{\vartheta, \eta\}$ is not good.

$Q = (2,1,1,2,0)$, (w) Figure 23: If $\zeta \in \{(M)_{3,7}, (M)_{6,4}\} \cap C_2$, then $\zeta = (M)_{2,6}$, and if $\eta \in \{(M)_{5,7}, (M)_{6,5}\} \cap C_2$, then $\eta = (M)_{2,3}$. Therefore, at least two positions in $\{(3,7),(5,7),(6,4),(6,5)\}$ are occupied by 3+colours. Since $c_{3+}(7) \leq 4$, at most one position in $\{(3,7),(5,7)\}$ and at least one position in $\{(6,4),(6,5)\}$ is occupied by a 3+colour. Further, from $r_{3+}(6) = 2$ it follows that exactly one position in $\{(6,4),(6,5)\}$ and in $\{(3,7),(5,7)\}$ as well is occupied by a 3+colour. Consequently, by Claim 3.6, $(M)_{2,7}$, $(M)_{6,1}$ and $(M)_{6,2}$ are three distinct 2-colours; this, however, leads to a contradiction, because if $\vartheta \in \{(M)_{2,7}, (M)_{6,1}, (M)_{6,2}\} \cap C_2$, then necessarily $\vartheta \in \{(M)_{3,6}, (M)_{5,3}\}$.

If $Q = (1,1,1,3,0)$, then we have (w) $\{\gamma, \delta, \varepsilon\} \cap \mathbb{R}(6) = \emptyset$. If a position in $\{6\} \times ([1,5] \cup \{7\})$ is occupied by a 2-colour ζ, then $\zeta = (M)_{2,6}$, which yields $r_{3+}(6) \geq 5$, a contradiction.

If $Q = (1,1,1,2,1)$, then the situation is (w) described by Figure 24. If a 2-colour ζ is in $\{6\} \times [1,2]$, then $\zeta = (M)_{5,6}$, hence $r_{3+}(6) \geq 3$.

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & * & * \\ \beta & \alpha & . & . & . & . & . \\ . & . & \delta & . & . & . & * \\ . & . & . & \varepsilon & . & . & . \\ . & . & . & . & . & \gamma & . \\ . & . & . & . & . & . & * \end{pmatrix}$$

Fig. 22.

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & * & * \\ \beta & \alpha & . & . & . & . & . \\ . & . & \delta & . & . & . & . \\ . & . & . & . & . & \gamma & * \\ . & . & . & . & . & \varepsilon & . \\ . & . & . & . & . & . & * \end{pmatrix}$$

Fig. 23.

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon & * & * \\ \beta & \alpha & . & . & . & . & . \\ . & . & * & . & . & \gamma & . \\ . & . & . & * & . & \delta & . \\ . & . & . & . & . & . & \varepsilon \\ . & . & * & * & . & . & . \end{pmatrix}$$

Fig. 24.

If $Q \in \mathcal{Q}(5,2)$, then we have $\sum_{j=1}^{7} q_j = 10$. Let $J = \{j \in [1,7] : q_j \geq 2\}$. In the case $|J| \leq 3$ realise that any colour $\zeta \in C_2 \setminus \{\alpha,\beta,\gamma,\delta,\varepsilon\}$ requires existence of a *sufficient* pair $(i,j) \in [2,6] \times [1,7]$, i.e., one satisfying $g(i,j,\{\alpha,\beta,\gamma,\delta,\varepsilon\}) \geq 3$. If (i,j) is a sufficient pair, then necessarily $j \in J$. Moreover, given $j \in J$, the number of sufficient pairs (i,j) is at most three. This is certainly true if $q_j = 3$. On the other hand, if $q_j = 2$ and $(M)_{k,l} = (M)_{1,j}$ with $k \neq 1$, then, by Claim 3.7 and the fact that $p = 2$, $(M)_{k,j} \notin \{\alpha,\beta,\gamma,\delta,\varepsilon\}$, which means that $g(k,j,\{\alpha,\beta,\gamma,\delta,\varepsilon\}) = 2$, and there are at most three i's such that the pair (i,j) is sufficient. Therefore, $c_2 \leq 5 + 3 \cdot 3 = 14$, which contradicts Claim 3.2.3.

So, we have $|J| \geq 4$. If $q_1 = 3$, then

$$10 = \sum_{j=1}^{7} q_j \geq 3 + 3 \cdot 2 + 1 \cdot 1 = 10,$$

hence $q_2 = q_3 = q_4 = 2$, $q_5 = 1$ and $q_6 = q_7 = 0$. If $\zeta = (M)_{i,j} \in C_2$ with $(i,j) \in [2,6] \times [6,7]$, then, since $g(i,j,\{\alpha,\beta,\gamma,\delta,\varepsilon\}) = 1$, we have $\zeta \in \mathbb{C}_2(1) \setminus \{\alpha,\beta,\gamma,\delta,\varepsilon\}$. Thus

$$c_{3+}(6) + c_{3+}(7) \geq 2 + (10 - 3) = 9,$$

and there is $j \in [6,7]$ with $c_{3+}(j) \geq 5$, a contradiction.

If $q_1 \leq 2$, then

$$g(i,j,\{\alpha,\beta,\gamma,\delta,\varepsilon\}) \leq q_j + 1 \leq q_1 + 1 \leq 3$$

for every $(i,j) \in [2,6] \times [1,5]$, hence $g(k,l,\{\alpha,\beta,\gamma,\delta,\varepsilon\}) \geq 2$ whenever $(k,l) \in [2,6] \times [6,7]$ and $(M)_{k,l} \in C_2$, which implies $q_l \geq 1$, $l = 6,7$. As a consequence, then

$$10 = \sum_{j=1}^{7} q_j \geq 2|J| + (7 - |J|) = |J| + 7 \geq 11,$$

a final contradiction proving Theorem 1.3.

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