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# Probabilistic modeling of mechanical cantilever oscillator fluctuation

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## ABSTRACT

The dynamics of oscillations of cantilever-type oscillators have been studied in this work. The probabilistic model of a cantilever is explained. Dependencies for the probability density, mean mathematical expectation, and standard deviation were developed. A fluctuation model of a cantilever was constructed, and the dependencies for the probability density for a direct problem were developed. The modeling of the mathematical expectation and phase variance for the inverse problem was performed.

**Keywords:** oscillator, atomic force microscope, probability density, mathematical expectation, variance

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## 1. Introduction

Mechanical oscillators of the cantilever type in the form of a microconsole with one free end have become widely used due to their high quality factor. Their use has greatly simplified the practical implementation of atomic force microscopy and significantly increased the sensitivity of analyzers in biosensors (Voigtländer 2015); also, the small value of the inertial mass of the cantilever allows us to significantly increase the resonant frequency of sensor fluctuation (Battiston et al. 2001).

Despite the fact that many works have been devoted to studying the dynamics of cantilever oscillations systems, the corresponding task remains relevant in the future (Pietrzakowski 2002; Romaszko et al. 2015). Sensor probes are exposed to random external actions, so the measured parameters require statistical and probabilistic averaging. So far, this problem has barely been solved or has been solved for partial cases (Haran Dr. n.d.; Zibenko, Tarasevich 2016).

## 2. Substantiation of probabilistic cantilever model

Let us omit the details of the substantiation of the cantilever oscillation dynamics under a force that is applied to the free end (Fig. 1) (Berman, Chumak 2007).

To reveal the main essence of the work, we assume that the unfixed end of a cantilever fluctuates as a system with concentrated parameters – inertia is expressed by the value of oscillator mass  $m$ , and elasticity is a parameter of rigidity  $\mu$ .

We assume that integral probability  $dF(t \leq T \leq t + dt) = f(t)dt$  is proportional to the width of interval  $dt$ :  $dF = \kappa \cdot dt$ , where constant coefficient  $\kappa$  can be determined from rationing condition  $\int_0^{2\pi/\omega_0} f(t)dt = \kappa \int_0^{2\pi/\omega_0} dt = 1$ , whence  $\kappa = \omega_0/2\pi$ . Therefore, the probability density function will be  $f(t) = 2\omega_0/\pi$ .

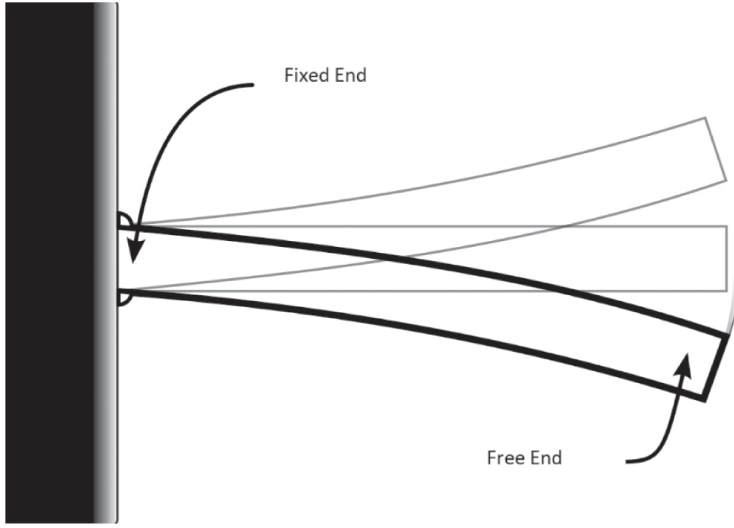


Fig. 1. Cantilever scheme

This means that random variable is evenly distributed. Its probability in interval  $[t_1, t_2]$  is  $F(t_1 \leq T \leq t_2) = \int_{t_1}^{t_2} f(t) dt = F(t_2) - F(t_1) = \omega_0 / 2\pi \cdot (t_2 - t_1)$ . The average  $\bar{T}$  for period  $2\pi/\omega_0$  is:

$$\bar{T} = \int_{-\infty}^{\infty} t \cdot f_T(t) \cdot dt = \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} t dt = \frac{\omega_0}{4\pi} \left( \frac{2\pi}{\omega_0} \right)^2 = \frac{\pi}{\omega_0}$$

and the mean  $\overline{T^2}$  of the square is:

$$\overline{T^2} = \int_{-\infty}^{\infty} t^2 \cdot f_T(t) \cdot dt = \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} t^2 \cdot dt = \frac{\omega_0}{6\pi} \left( \frac{2\pi}{\omega_0} \right)^3 = \frac{1}{3} \left( \frac{2\pi}{\omega_0} \right)^2$$

Then, the variance of the random variable is:

$$D_T = \overline{T^2} - (\bar{T})^2 = \frac{1}{3} \left( \frac{2\pi}{\omega_0} \right)^2 - \frac{1}{4} \left( \frac{2\pi}{\omega_0} \right)^2 = \frac{1}{12} \left( \frac{2\pi}{\omega_0} \right)^2 = \frac{1}{3} \left( \frac{\pi}{\omega_0} \right)^2 > 0 \quad (1)$$

and its standard deviation is:

$$\sigma_T = \frac{1}{\sqrt{3}} \frac{\pi}{\omega_0} \quad (2)$$

that is, the condition of non-dispersion is satisfied.

Under initial conditions  $y|_{t=0} = 0$ , the change in amplitude in a single period is described by function:

$$y = y_0 \cos \omega_0 t \tag{3}$$

Possible values of transformed random variation  $Y = y_0 \cos \omega_0 T$  are placed in interval  $Y \in [-y_0, y_0]$ , beyond which probability density  $f_Y(y) = 0$ . This interval of Equation (3) has two roots.

To justify the probability density  $f_Y(y)$  for values  $Y \in [-y_0, y_0]$ , let us set the explicit form of Inverse Function (3):

$$t = \frac{1}{\omega_0} \cos^{-1} \frac{y}{y_0}$$

and a derivation of the inverse:

$$\frac{d}{dy} \cos^{-1} y = -\frac{1}{\pi \sqrt{y_0^2 - y^2}} \tag{4}$$

Function  $y = g(y) = y_0 \cos \omega_0 t$  is non-monotonic for both the original and converted random variable, and the connection of cumulative probabilities is true:

$$\begin{aligned} F_Y(y) &= P[Y \leq y] = P[y_0 \cos \omega_0 t \leq y] = \\ &= P\left[\cos^{-1} \frac{y}{y_0} \leq 2\pi - \cos^{-1} \frac{y}{y_0}\right] = 1 - \frac{1}{\pi} \cos^{-1} \frac{y}{y_0} \end{aligned} \tag{5}$$

where the derivative of (4) gives probability density  $f_Y(y)$ :

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \left(1 - \frac{1}{\pi} \cos^{-1} \frac{y}{y_0}\right) = \frac{1}{\pi \sqrt{y_0^2 - y^2}} \tag{6}$$

The graph of Function (6) goes to infinity at  $y \rightarrow \pm y_0$  due to the slowdown of function  $y = \cos \omega_0 t$  in the vicinity of extreme points  $\omega_0 t \rightarrow 0, \pi$ . Dependence (6) is due to the fact that, under a relatively large sample size (the number of measurements) where value  $t$  from interval  $2\pi/\omega_0$  is randomly selecting each time, size  $\cos \omega_0 t$  more likely to take a value that is closer to  $\pm 1$  than it is to zero (Consortini 2000).

In absolute value, the amplitude varies in interval  $[0, y_0]$ . Therefore, the average  $\bar{Y}$  (mathematical expectation) is:

$$\bar{Y} = \int_{-\infty}^{\infty} y \cdot f_Y(y) \cdot dy = \frac{1}{\pi} \int_0^{y_0} \frac{y}{\sqrt{y_0^2 - y^2}} dy = \frac{1}{\pi} \int_0^{y_0} \frac{d(y_0^2 - y^2)}{\sqrt{y_0^2 - y^2}} = -\frac{1}{\pi} \sqrt{y_0^2 - y^2} \Big|_0^{y_0} = \frac{y_0}{\pi}$$

and the mean square (intensity) is:

$$\begin{aligned} \overline{Y^2} &= \int_{-\infty}^{\infty} y^2 \cdot f_Y(y) \cdot dy = \frac{1}{\pi} \int_0^{y_0} \frac{y^2}{\sqrt{y_0^2 - y^2}} dy = \\ &= \frac{1}{\pi} \int_0^{y_0} \frac{(y_0^2 - y^2)}{\sqrt{y_0^2 - y^2}} dy - \frac{1}{\pi} \int_0^{y_0} \frac{y_0^2}{\sqrt{y_0^2 - y^2}} dy = -\frac{1}{\pi} \int_0^{y_0} \sqrt{y_0^2 - y^2} dy + \frac{2y_0^2}{\pi} \int_0^{y_0} \frac{dy}{\sqrt{y_0^2 - y^2}} \end{aligned}$$

We calculate first integral  $-1/2 \int_0^{y_0} \sqrt{y_0^2 - y^2} dy$  in stages:

$$\begin{aligned} \int_0^{y_0} \sqrt{y_0^2 - y^2} dy &= \left| \begin{array}{l} u = \sqrt{y_0^2 - y^2} \quad dv = dy \\ du = -\frac{y \cdot dy}{\sqrt{y_0^2 - y^2}} \quad v = y \end{array} \right| = y\sqrt{y_0^2 - y^2} \Big|_0^{y_0} - \int_0^{y_0} \frac{-y^2 dy}{\sqrt{y_0^2 - y^2}} = \\ &= \int_0^{y_0} \frac{(y_0^2 - y^2 - y_0^2) dy}{\sqrt{y_0^2 - y^2}} = -\int_0^{y_0} \sqrt{y_0^2 - y^2} dy + y_0^2 \int_0^{y_0} \frac{dy}{\sqrt{y_0^2 - y^2}} \end{aligned}$$

Thus,  $\int_0^{y_0} \sqrt{y_0^2 - y^2} dy = y_0^2/2 \int_0^{y_0} dy/\sqrt{y_0^2 - y^2}$ , and the variance of the amplitude is:

$$\begin{aligned} D_Y &= \overline{Y^2} - (\bar{Y})^2 = \frac{y_0^2}{\pi} \int_0^{y_0} \frac{dy}{\sqrt{y_0^2 - y^2}} - \left(\frac{y_0}{\pi}\right)^2 = \frac{y_0^2}{\pi} \arcsin \frac{y}{y_0} \Big|_0^{y_0} - \left(\frac{y_0}{\pi}\right)^2 = \\ &= \frac{y_0^2}{2} - \left(\frac{y_0}{\pi}\right)^2 = \frac{y_0^2}{2} \left(1 - \frac{2}{\pi}\right) \end{aligned}$$

That is, the condition of non-dispersion is satisfied. The standard deviation of the amplitude is:

$$\sigma_Y = \frac{y_0}{\sqrt{2}} \sqrt{1 - \frac{2}{\pi}}$$

### 3. Fluctuation cantilever model

#### 3.1. Equal distribution of fluctuation phase (direct task)

Let us build a model of average amplitude when the  $t \omega_0 t$  argument function:

$$x = y_0 \cos(\omega_0 t + \psi) \tag{7}$$

phase  $\psi$  gets random changes. Let us assume that phase  $\psi$  is distributed evenly with probability density  $f_\psi(\psi) = 1/2\pi$ . Then, for Transformation (7), the inverse function and its derivation is  $\psi = g^{-1}(x) = \cos^{-1} x$  and  $(d/dz)\cos^{-1} x = -1/(\pi\sqrt{x_0^2 - x^2})$ . Function (7) is ambiguous and has two roots:  $\psi_1 = \cos^{-1}(x/y_0) - \omega_0 t$ , and  $\psi_2 = 2\pi - \psi_1$ . Therefore, the density of probabilities  $f_X(x)$  is:

$$f_X(x) = \frac{f_\psi(\psi_1)}{|\pi\sqrt{y_0^2 - x^2}|} + \frac{f_\psi(\psi_2)}{|\pi\sqrt{y_0^2 - x^2}|} = \frac{1}{\pi^2} \frac{1}{\sqrt{y_0^2 - x^2}}$$

The essence of averaging the  $\bar{X}$  and  $\overline{X^2}$  amplitude of fluctuation with a random phase probability density  $f_\psi(\psi)$  is the averaging of trigonometric functions  $\overline{\cos \Psi}$  and  $\overline{\cos^2 \Psi}$ ; that is, the averages of  $\bar{X}$  and  $\overline{X^2}$  are:

$$\bar{X} = y_0 \overline{\cos \psi} = y_0 \int_{-\infty}^{\infty} \cos \psi f_\psi(\psi) d\psi = -\frac{y_0}{2\pi} \sin \psi \Big|_0^{2\pi} = 0$$

$$\overline{X^2} = y_0^2 \overline{\cos^2 \psi} = y_0^2 \int_{-\infty}^{\infty} \cos^2 \psi f_\psi(\psi) d\psi = \frac{y_0^2}{4\pi} \int_0^{2\pi} (1 + \cos 2\psi) d\psi = \frac{y_0^2}{2}$$

Amplitude variance is calculated by the formula:

$$D_X = \overline{X^2} - (\bar{X})^2 = \frac{y_0^2}{2} \Rightarrow \sigma_X = \frac{y_0}{\sqrt{2}}$$

### 3.2. Rated phase fluctuation distribution

The case when random phase changes are subject to rated distribution with mathematical expectation  $m_\Phi$  and variance  $\sigma_\Phi^2$  ( $\phi \in N(m_\Phi, \sigma_\Phi^2)$ ) are more interesting from a practical point of view:

$$f_\Phi(\phi) = \frac{1}{\sqrt{2\pi}\sigma_\Phi} \exp\left(-\frac{(\phi - m_\Phi)^2}{2\sigma_\Phi^2}\right) \quad (8)$$

To convert  $Z = y_0 \cos \Phi$ , the inverse function and its derivation is  $\phi = g^{-1}(z) = \cos^{-1} z (d/dz) \cos^{-1} z = -1/(\pi\sqrt{y_0^2 - z^2})$ . Functions  $z = \cos\phi$  and  $z^2 = \cos^2\phi$  are ambiguous with periods  $2\pi/\omega_0, \pi/\omega_0$  and have two roots:  $\phi_1 = \cos^{-1} z$  and  $\phi_2 = 2\pi - \phi_1$ ; with  $\phi_3 = \cos^{-1}(2z^2 - 1)$  and  $\phi_4 = \pi - \phi_3$ , we obtain that probability density  $f_Z(z)$  is:

$$\begin{aligned} f_Z(z) &= \frac{1}{\sigma_\Phi^2 \sqrt{2\pi}} \begin{cases} \left| \frac{d}{dz} \cos^{-1} z \right| \cdot \exp\left(-\frac{1}{2\sigma_\Phi^2} [\cos^{-1} z - m_\Phi]^2\right) & \text{if } \cos^{-1} z \in [0, y_0], z \in [-\pi, 0) \\ \left| \frac{d}{dz} \cos^{-1} z \right| \cdot \exp\left(-\frac{1}{2\sigma_\Phi^2} [\cos^{-1} z - m_\Phi]^2\right) & \text{if } \cos^{-1} z \in [0, y_0], z \in [0, \pi] \end{cases} = \\ &= \frac{1}{\sigma_\Phi^2 \sqrt{2\pi}} \frac{1}{\sqrt{1-z^2}} \cdot \exp\left(-\frac{1}{2\sigma_\Phi^2} [\cos^{-1} \phi - m_\Phi]^2\right) \quad \text{if } \cos^{-1} \phi \in [0, 1], z \in [-\pi, \pi] \end{aligned} \quad (9)$$

The average in this case are:

$$\begin{aligned} \bar{Z} &= I_1 + I_2 = \frac{y_0}{2} \exp\left(-\frac{\sigma_\Phi^2}{2}\right) [\exp(im_\Phi) + \exp(-im_\Phi)] = y_0 \exp\left(-\frac{\sigma_\Phi^2}{2}\right) \cos m_\Phi, \\ \bar{Z}^2 &= I_1 + I_2 + I_3 = \frac{y_0^2}{2} + \frac{y_0^2}{2} \exp(-2\sigma_\Phi^2) \cos 2m_\Phi = \frac{y_0^2}{2} \left(1 + e^{-2\sigma_\Phi^2} \cos 2m_\Phi\right) \end{aligned} \quad (10)$$

and the variance of the amplitude is:

$$D_Z = \bar{Z}^2 - (\bar{Z})^2 = \frac{y_0^2}{2} \left(1 + e^{2\sigma_\Phi^2} \cos 2m_\Phi\right) + y_0^2 \exp(-\sigma_\Phi^2) \cos^2 m_\Phi$$

### 3.3. Mathematical model expectation and phase variance by equal amplitude distribution

Suppose that a sample of random amplitude values is obtained by measurements  $s = y_0 \sin \theta$  with equal distribution  $f_S(s) = 1/y_0$ . Phase  $\theta$  is defined as the inverse of the  $S = y_0 \sin \Theta$  transformation of random variable  $S/y_0$ :

$$\Theta = \sin^{-1} \frac{S}{y_0} \tag{11}$$

Function  $\theta = g(s/y_0) = \sin^{-1}(s/y_0)$  is ambiguous and has two roots in interval  $[-y_0, y_0]$ ;  $S_1/y_0 = \sin \Theta$  and  $S_2/y_0 = -\sin \Theta$ ; inverted  $s/y_0 = g^{-1}(\theta) = \sin \theta$  and derived from it:

$$\left| \frac{d\left(\frac{s}{y_0}\right)}{d\theta} \right| = \left| \frac{d[\sin \theta]}{d\theta} \right| = |\cos \theta| \tag{12}$$

The connection is true for cumulative probabilities in the function monotonicity of random variable transformation:

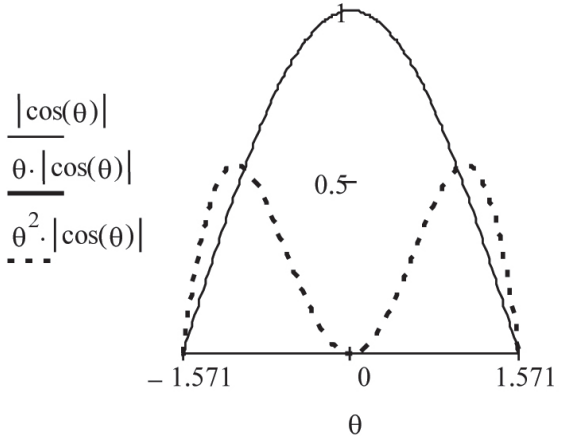
$$F_{\Theta}(\theta) = P\left[\left(\sin^{-1} \frac{S}{y_0}\right) \leq \theta\right] = P\left[\frac{S_2}{y_0} \leq \frac{S}{y_0} \leq \frac{S_1}{y_0}\right] = F_S(+\sin \theta) - F_S(-\sin \theta) \tag{13}$$

and we get function by differentiation  $f_{\Theta}(\theta) : f_{\Theta}(\theta) = (2/y_0)y_0 |\cos \theta| = 2 |\cos \theta|$ .

Let us set the boundaries of integration. Since the limits of parameter change  $s$  are  $-y_0 \leq s \leq y_0$  in Formula (10), then  $-1 \leq (s/y_0) \leq 1$ ; and according to equation  $\theta = \sin^{-1}(s/y_0)$  for phase  $\theta$ , the limits of integration are  $-\pi/2 \leq \theta \leq \pi/2$  (Fig. 2).

Then, the average  $\bar{\Theta}$  is obtained by integrating the parts of the integral:

$$\begin{aligned} \bar{\Theta} &= 2 \int_{-\pi/2}^{\pi/2} \theta \cos \theta d\theta = \left| \begin{array}{ll} u = \theta & dv = \cos \theta d\theta \\ du = d\theta & v = \sin \theta \end{array} \right| = \\ &= 2 \theta \sin \theta \Big|_{-\pi/2}^{\pi/2} + 2 \int_{-\pi/2}^{\pi/2} \sin \theta d\theta = 2\pi \end{aligned} \tag{14}$$



**Fig. 2.** Graphs of functions  $f_{\Theta}(\theta)$  and subintegral expressions in Formulas (14) and (15)

The mean square of Integration (3) is:

$$\begin{aligned}
 \overline{\Theta^2} &= 2 \int_{-\pi/2}^{\pi/2} \theta^2 \cos \theta d\theta = \left. \begin{array}{l} u = \theta^2 \quad dv = \sin \theta d\theta \\ du = 2\theta d\theta \quad v = -\cos \theta \end{array} \right| = \\
 &= \theta^2 \sin \theta \Big|_{-\pi/2}^{\pi/2} + 2 \int_{-\pi/2}^{\pi/2} \theta \sin \theta d\theta = \pi^2 + 2 \left. \begin{array}{l} u = \theta \quad dv = \sin \theta d\theta \\ du = d\theta \quad v = -\cos \theta \end{array} \right| = \quad (15) \\
 &= \pi^2 + 2\pi - 2 \int_{-\pi/2}^{\pi/2} \cos \theta d\theta = \pi^2 + 2\pi - 4
 \end{aligned}$$

### 3.4. Rated $N(m_Q, \sigma_Q^2)$ distribution of random amplitude values

For law of fluctuations  $q = y_0 \sin(\omega_0 t + w)$ , the inverse transformation of random variable  $Q/y_0$  has similar relevant patterns to those that are discussed in this paragraph. Then, the distribution function  $f_W(w)$  of the probability density of converted random variable  $Q/y_0$  is the following (Consortini 2000; Gradshteyn, Ryzhik 1963):

$$f_{\Theta}(\theta) = \frac{y_0}{\sqrt{2\pi\sigma_Q^2}} \cos(\theta + \Theta_0) \exp\left(-\frac{[y_0 \cos(\theta + \Theta_0) - m_Q]^2}{2\sigma_Q^2}\right) \quad (16)$$

Then, the average of  $\bar{\Theta}$  i  $\bar{\Theta}^2$  is:

$$\bar{\Theta} = 2 \int_{-\infty}^{+\infty} \theta \cos(\theta + \Theta_0) d\theta = 4 \int_0^{+\infty} \theta \cos(\theta + \Theta_0) d\theta$$

$$\bar{\Theta}^2 = 2 \int_{-\infty}^{+\infty} \theta^2 \cos(\theta + \Theta_0) d\theta = 4 \int_0^{+\infty} \theta^2 \cos(\theta + \Theta_0) d\theta$$
(17)

## 4. Conclusions

The dynamics of the oscillations of cantilever-type oscillators has been studied. The probabilistic model of a cantilever is explained. The dependencies for the probability density, mean mathematical expectation, and standard deviation were developed. The fluctuation model of the cantilever was constructed, and the dependencies for the probability density for the direct problem were developed. The modeling of the mathematical expectation and phase variance for the inverse problem was performed.

## REFERENCES

- Battiston F.M., Ramseyer J.-P., Lang H.P., Baller M.K., Gerber Ch., Gimzewski J.K., Meyer E., Güntherodt H.-J., 2001, *A chemical sensor based on a microfabricated cantilever array with simultaneous resonance frequency and bending readout*, Sensor and Actuators B: Chemical, 77(1–2), 122–131.
- Berman G.P., Chumak A.A., 2007, *Intrinsic dissipation in cantilevers*, AirXiv: 0709.2582v1, <https://arxiv.org/abs/0709.2582> [17.09.2007].
- Consortini A., 2000, *Should optics students know statistics?* In: Javier Sanchez-Mondragon J. (Ed.), *Sixth International Conference on Education and Training in Optics and Photonics*, Proc. SPIE, 3831, 47–55.
- Gradshteyn I.S., Ryzhik I.M., 1963, *Tablitsy integralov, summ, ryadov i proizvedeniy*, Fizmatgiz, Moskva [Градштейн И. С., Рыжик И.М., 1963, *Таблицы интегралов, сумм, рядов и произведений*, Физматгиз, Москва].
- Haran Dr. *Second-order Systems: Vibrating Cantilever Beams*, ME 3504 Process Monitoring & Control [online]: <https://www.yumpu.com/en/document/view/12369805/second-order-systems-vibrating-cantilever-beams> [10.09.2021].

- Pietrzakowski M., 2002, *Experiment on a cantilever beam control and theoretical approximation*, Journal of Theoretical and Applied Mechanics, 40(3), 667–686.
- Romaszko M., Sapiński, Sioma A., 2015, *Forced vibrations analysis of a cantilever beams using the vision method*, Journal of Theoretical and Applied Mechanics, 53(1), 243–254.
- Voigtländer B., 2015, *Scanning Probe Microscopy. Atomic Force Microscopy and Scanning Tunneling Microscopy*, Springer.
- Tsybenko S.O., Tarasevych Yu.Ya., 2016, *Ймовірнісні методи в механіці*, Natsional'nyy Tekhnichnyy Universytet, „Kharkivs'kyy Politekhnichnyy Instytut”, Kharkiv [Цибенко С.О.,Тарасевич Ю.Я., 2016, *Ймовірнісні методи в механіці*, Національний Технічний Університет „Харківський Політехнічний Інститут”, Харків].