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EXISTENCE OF THREE SOLUTIONS FOR IMPULSIVE MULTI-POINT BOUNDARY VALUE PROBLEMS

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Abstract. This paper is devoted to the study of the existence of at least three classical solutions for a second-order multi-point boundary value problem with impulsive effects. We use variational methods for smooth functionals defined on reflexive Banach spaces in order to achieve our results. Also by presenting an example, we ensure the applicability of our results.

Keywords: multi-point boundary value problem, impulsive condition, classical solution, variational method, three critical points theorem.

Mathematics Subject Classification: 34B10, 34B15, 34A37.

1. INTRODUCTION

In this paper, we consider the second-order ℓ-point boundary value problem with m-impulsive effects

$$\begin{cases} -(\phi_{p_i}(u_i'))' = \lambda F_{u_i}(t, u_1, \ldots, u_n) + \mu G_{u_i}(t, u_1, \ldots, u_n), \quad t \in (0,1) \setminus Q, \\ \Delta\phi_{p_i}(u_i'(t_j)) = I_{ij}(u_i(t_j)), \quad j = 1, \ldots, m, \\ u_i(0) = \sum_{k=1}^{\ell} a_k u_i(s_k), \quad u_i(1) = \sum_{k=1}^{\ell} b_k u_i(s_k) \end{cases} \tag{$\phi_{\lambda,\mu}$}$$

for $i = 1, \ldots, n$, where $Q = \{t_1, \ldots, t_m\}$, $p_i \in (1, \infty)$, $\phi_{p_i}(x) = |x|^{p_i-2}x$ for $i = 1, \ldots, n$, $\lambda > 0, \mu \geq 0$ are parameters, $m, n, \ell \in \mathbb{N}$, $0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = 1$, $0 < s_1 \leq s_2 \leq \ldots \leq s_\ell < 1$, $t_j \neq s_k$, $j = 1, \ldots, m$, $k = 1, \ldots, \ell$, $F, G : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ are measurable with respect to t, for every $(x_1, \ldots, x_n) \in \mathbb{R}^n$, continuously differentiable in $(x_1,\dots,x_n)$, for almost every $t\in[0,1]$ and satisfy the standard summability condition

$$\sup_{|\xi|\le\varrho_1}\left\{\max\{|F(\cdot,\xi)|,\ |G(\cdot,\xi)|,\ |F_{\xi_i}(\cdot,\xi)|,\ |G_{\xi_i}(\cdot,\xi)|,\ i=1,\dots,n\}\right\}\in \mathrm{L}^1([0,1]) \tag{1.1}$$

for any $\varrho_1>0$ with $\xi=(\xi_1,\dots,\xi_n)$ and $|\xi|=\sqrt{\sum_{i=1}^n\xi_i^2}$, F_{u_i} and G_{u_i} are the partial derivatives of F and G with respect to u_i, respectively,

$$\Delta\phi_{p_i}(u_i'(t_j))=\phi_{p_i}(u_i'(t_j^+))-\phi_{p_i}(u_i'(t_j^-)),$$

where $u_i'(t_j^+)$ and $u_i'(t_j^-)$ represent the right-hand limit and left-hand limit of $u_i'(t)$ at $t=t_j$, respectively, $I_{ij}\in \mathrm{C}(\mathbb{R},\mathbb{R})$, $i=1,\dots,n$, $j=1,\dots,m$ and $a_k,b_k\in\mathbb{R}$ for $k=1,\dots,\ell$.

The theory of multi-point boundary value problems for ordinary differential equations arise in different areas of applied mathematics and physics, especially in heat conduction [4, 5, 27, 29], the vibration of cables with nonuniform weights [35], and other problems in nonlinear elasticity [45]. The study of multi-point BVPs for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [26]. From then on, many authors studied more general nonlinear multi-point boundary value problems. Recently, the existence and multiplicity of positive solutions for nonlinear multi-point BVPs have received a great deal of attention, we refer the reader to [11, 12, 15, 16, 18–20, 23–25, 34] and the references therein.

On the other hand, impulsive differential equations serve as basic models to study the dynamics of processes that are subjected to sudden changes in their states. These kinds of processes naturally occur in control theory, biology, optimization theory, medicine, and so on (see [6, 10, 28, 36]). The theory of impulsive differential equations has recently received considerable attention, see [2, 3, 6, 30, 33, 39]. There has been increasing interest in the investigation for boundary value problems of nonlinear impulsive differential equations during the past few years, and many works have been published about the existence of solutions for second-order impulsive differential equations. There are some common techniques to approach these problems: Fixed point theorems [8, 9, 31], the method of upper and lower solutions [7], and topological degree theory [37]. In the last few years, variational methods and critical point theory have been used to determine the existence of solutions for impulsive differential equations under certain boundary conditions, see [1, 21, 43, 44, 46, 48] and the references therein. We note that the difficulties dealing with such problems are that their states are discontinuous. Therefore, the results of impulsive differential equations, especially for higher-order impulsive differential equations, are fewer in number than those for differential equations without impulses.

Moreover, some researchers have studied the existence and multiplicity of solutions for multi-point boundary value problems for second-order impulsive differential equations; we refer the reader to [13, 14, 32, 42] and the references therein. For example in [13], Feng and Pang used fixed-point index theory and a fixed-point theorem in the cone of strict set contraction operators to obtained some new results for the existence and multiplicity of positive solutions of a boundary value problem for second-order three-point nonlinear impulsive integrodifferential equation of mixed type in a real Banach space. In [32], Liu and Yu with the help of the coincidence degree continuation theorem, achieved a general result concerning the existence of solutions of m-point boundary value problems for second-order differential systems with impulses. They also give a definition of autonomous curvature bound set relative to this m-point boundary value problems, and by using this definition and the above existence theorem, obtained some simple existence conditions for solutions of these boundary value problems. Meiqiang and Dongxiu in [14], based on fixed point theory in a cone, discussed the existence of solutions for the m-point BVPs for second-order impulsive differential systems, and Thaiprayoon *et al.* in [42], introduced a new definition of impulsive conditions for boundary value problems of first-order impulsive differential equations with multi-point boundary conditions was introduced, and used the method of lower and upper solutions in reversed order coupled with the monotone iterative technique to obtain the extremal solutions of the boundary value problem.

Motivated by the above works, in this paper we are interested to investigate the existence of at least three nontrivial classical solutions for second-order ℓ-point boundary value problems with m-impulsive effects $(\phi_{\lambda,\mu})$ for appropriate values of the parameters λ and μ belonging to real intervals. Our approach uses variational methods and a three critical points theorem due to Ricceri [38].

Here, we state a special case of our main result.

Theorem 1.1. *Let $p_1, p_2 > 1$ such that either $\min\{p_1,p_2\} \geq 2$ or $\max\{p_1,p_2\} < 2$, $F : \mathbb{R}^2 \to \mathbb{R}$ be a C^1-function satisfying the condition*

$$\sup_{\sqrt{\xi_1^2+\xi_2^2}\leq\varrho_2} \max\{|F(\xi_1,\xi_2)|,\ |F_{\xi_1}(\xi_1,\xi_2)|,\ |F_{\xi_2}(\xi_1,\xi_2)|\} \in \mathrm{L}^1([0,1])$$

for any $\varrho_2 > 0$, $F(0,0) = 0$, $I_{11}, I_{12}, I_{21}, I_{22} \in \mathrm{C}(\mathbb{R},\mathbb{R})$ be nondecreasing functions such that $I_{ij}(0) = 0$ and $I_{ij}(s)s > 0$, $s \neq 0$ for $i = 1,2$, $j = 1,2$, $a_1, a_2, b_1, b_2 \in \mathbb{R}$ such that $a_1 + a_2 \neq 1$ and $b_1 + b_2 \neq 1$, $0 < t_1 < t_2 < 1$, $0 < s_1 \leq s_2 < 1$, and $t_j \neq s_k$, $j = 1,2$, $k = 1,2$. Assume that

$$\max\left\{\limsup_{(u_1,u_2)\to(0,0)} \frac{F(u_1,u_2)}{|u_1|^{p_1}+|u_2|^{p_2}},\ \limsup_{|u|\to\infty} \frac{F(u_1,u_2)}{|u_1|^{p_1}+|u_2|^{p_2}}\right\} < 0$$

and

$$\sup_{u\in\widehat{E}_1\times\widehat{E}_2} \frac{\int_0^1 F(u_1(t),u_2(t))\mathrm{d}t}{\sum_{i=1}^2 \frac{\|u_i\|_{p_i}^{p_i}}{p_i} + \sum_{i=1}^2\sum_{j=1}^2 \int_0^{u_i(t_j)} I_{ij}(\zeta)\mathrm{d}\zeta} > 0,$$

where

$$\widehat{E}_i = \left\{\xi \in \mathrm{W}^{1,p_i}([0,1]) :\ \xi(0) = a_1\xi(s_1)+a_2\xi(s_2),\ \xi(1) = b_1\xi(s_1)+b_2\xi(s_2)\right\},\ i=1,2.$$

Then, for each compact interval $[c,d] \subset (\tilde{\lambda}, \infty)$*, where*

$$\tilde{\lambda} = \inf_{u \in \widehat{E}_1 \times \widehat{E}_2} \left\{ \frac{\sum_{i=1}^{2} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} + \sum_{i=1}^{2} \sum_{j=1}^{2} \int\limits_0^{u_i(t_j)} I_{ij}(\zeta)\mathrm{d}\zeta}{\int\limits_0^1 F(u_1(t), u_2(t))\mathrm{d}t} : \int\limits_0^1 F(u_1(t), u_2(t))\mathrm{d}t > 0 \right\},$$

there exists $R > 0$ *with the following property: for every* $\lambda \in [c,d]$ *and for every* $G \in \mathrm{C}^1(\mathbb{R}^2, \mathbb{R})$ *such that*

$$\sup_{\sqrt{\xi_1^2 + \xi_2^2} \leq \varrho_3} \max\{|G(\xi_1, \xi_2)|, \ |G_{\xi_1}(\xi_1, \xi_2)|, \ |G_{\xi_2}(\xi_1, \xi_2)|\} \in \mathrm{L}^1([0,1])$$

for any $\varrho_3 > 0$ *with* $G(0,0) = 0$*, there exists* $\gamma > 0$ *such that, for each* $\mu \in [0, \gamma]$*, the system*

$$\begin{cases} -|u_i'(t)|^{p_i-2} u_i''(t) = \lambda F_{u_i}(u_1, u_2) + \mu G_{u_i}(u_1, u_2), & t \in (0,1) \setminus \{t_1, t_2\}, \\ |u_i'(t_j^+)|^{p_i-2} u_i''(t_j^+) - |u_i'(t_j^-)|^{p_i-2} u_i''(t_j^-) = I_{ij}(u_i(t_j)), & j = 1, 2, \\ u_i(0) = a_1 u_i(s_1) + a_2 u_i(s_2), \quad u_i(1) = b_1 u_i(s_1) + b_2 u_i(s_2) \end{cases}$$

for $i = 1, 2$*, has at least three classical solutions whose norms in the space* $\widehat{E}_1 \times \widehat{E}_2$ *are less than* R*.*

2. PRELIMINARIES

In this section, we will introduce some notations, definitions and preliminary facts which are used throughout this paper.

To construct appropriate function spaces and apply critical point theory in order to investigate the existence of solutions for system $(\phi_{\lambda,\mu})$, we introduce the following basic notations and results which will be used in the proofs of our main results.

Throughout this article, we let E be the Cartesian product of n spaces

$$E_i = \left\{ \xi \in \mathrm{W}^{1,p_i}([0,1]) : \xi(0) = \sum_{k=1}^{\ell} a_k \xi(s_k), \ \xi(1) = \sum_{k=1}^{\ell} b_k \xi(s_k) \right\}$$

for $i = 1, \ldots, n$, i.e., $E = E_1 \times \ldots \times E_n$, endowed with the norm

$$\|u\| = \|(u_1, \ldots, u_n)\| = \sum_{i=1}^{n} \|u_i\|_{p_i},$$

where

$$\|u_i\|_{p_i} = \left(\int\limits_0^1 |u_i'(t)|^{p_i} \, \mathrm{d}t \right)^{\frac{1}{p_i}}$$

for $i = 1, \ldots, n$. Then E is a reflexive real Banach space.

In this paper, we assume throughout, and without further mention, that the following conditions hold:

(H1) either $\underline{p} \geq 2$ or $\overline{p} < 2$, where $\underline{p} = \min\{p_1, \ldots, p_n\}$ and $\overline{p} = \max\{p_1, \ldots, p_n\}$;
(H2) $\sum_{k=1}^{\ell} a_k \neq 1$ and $\sum_{k=1}^{\ell} b_k \neq 1$.

Let

$$c = \max_{1\leq i\leq n} \left\{ \sup_{u_i \in E_i \setminus \{0\}} \frac{\max_{t\in[0,1]} |u_i(t)|^{p_i}}{\|u_i\|_{p_i}^{p_i}} \right\}. \tag{2.1}$$

Since $p_i > 1$ for $i = 1, \ldots, n$, the embedding $E = E_1 \times \ldots \times E_n \hookrightarrow \left(\mathrm{C}^0([0,1])\right)^n$ is compact, and so $c < \infty$. Moreover, by $(H2)$, from [11, Lemma 3.1], we have

$$\sup_{v\in E_i\setminus\{0\}} \frac{\max_{t\in[0,1]} |v(t)|^{p_i}}{\|v\|_{p_i}^{p_i}} \leq \frac{1}{2} \left(1 + \frac{\sum_{k=1}^{\ell} |a_k|}{|1 - \sum_{k=1}^{\ell} a_k|} + \frac{\sum_{k=1}^{\ell} |b_k|}{|1 - \sum_{k=1}^{\ell} b_k|} \right) \tag{2.2}$$

for $i = 1, \ldots, n$.

By a classical solution of $(\phi_{\lambda,\mu})$, we mean a function $u = (u_1, \ldots, u_n)$ such that, for $i = 1, \ldots, n$, the left-hand side limit and the right-hand side limit of the derivative u_i in t_j must exist and must be finite, $u_i \in \mathrm{C}^1([0,1] \setminus Q)$, $\phi_{p_i}(u_i') \in \mathrm{C}^1([0,1] \setminus Q)$, and u_i satisfies $(\phi_{\lambda,\mu})$. We say that a function $u = (u_1, \ldots, u_n) \in E$ is a weak solution of $(\phi_{\lambda,\mu})$ if

$$\begin{aligned}
&\int_0^1 \left(\sum_{i=1}^{n} \phi_{p_i}(u_i'(t)) v_i'(t) \right) \mathrm{d}t + \sum_{i=1}^{n} \sum_{j=1}^{m} I_{ij}(u_i(t_j)) v_i(t_j) \\
&- \lambda \int_0^1 \sum_{i=1}^{n} F_{u_i}(t, u_1(t), \ldots, u_n(t)) v_i(t) \mathrm{d}t - \mu \int_0^1 \sum_{i=1}^{n} G_{u_i}(t, u_1(t), \ldots, u_n(t)) v_i(t) \mathrm{d}t = 0
\end{aligned}$$

for any $v = (v_1, \ldots, v_n) \in \mathrm{W}_0^{1,p_1}([0,1]) \times \mathrm{W}_0^{1,p_2}([0,1]) \times \ldots \times \mathrm{W}_0^{1,p_n}([0,1])$.

Let $\phi_{p_i}^{-1}$ denote the inverse of ϕ_{p_i} for every $i = 1, \ldots, n$. Then $\phi_{p_i}^{-1} = \phi_{q_i}$, where $\frac{1}{p_i} + \frac{1}{q_i} = 1$. It is clear that ϕ_{p_i} is increasing on $\mathbb{R}$,

$$\lim_{x\to-\infty} \phi_{p_i}(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} \phi_{p_i}(x) = \infty. \tag{2.3}$$

Lemma 2.1. *For fixed $\lambda, \mu \in \mathbb{R}$, $u = (u_1, \ldots, u_n) \in (\mathrm{C}(t_j, t_{j+1}))^n$, $j = 0, 1, \ldots, m$, define $\alpha_{ij}(x; u) : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$, $j = 0, 1, \ldots, m$ by*

$$\begin{aligned}
&\alpha_{ij}(x; u) \\
&= \int_{t_j}^{t_{j+1}} \phi_{p_i}^{-1} \left(x - \lambda \int_0^{\delta} F_{u_i}(t, u_1(t), \ldots, u_n(t)) \mathrm{d}t - \mu \int_0^{\delta} G_{u_i}(t, u_1(t), \ldots, u_n(t)) \mathrm{d}t \right) \mathrm{d}\delta \\
&\quad + \sum_{k=1}^{\ell} a_k u_k(s_k) - \sum_{k=1}^{\ell} b_k u_k(s_k).
\end{aligned}$$

Then, the equation

$$\alpha_{ij}(x;u)=0 \tag{2.4}$$

has a unique solution $x_{u,i,j}$.

Proof. Taking (2.3) into account, we have

$$\lim_{x\to-\infty}\alpha_{ij}(x,u)=-\infty \quad \text{and} \quad \lim_{x\to\infty}\alpha_{ij}(x,u)=\infty.$$

Since $\alpha_{ij}(\cdot,u)$ is continuous and strictly increasing on $\mathbb{R}$, the conclusion follows for $i=1,\dots,n$ and $j=1,\dots,m$. □

Lemma 2.2. *The function* $u=(u_1,\dots,u_n)$ *is a solution of the system* $(\phi_{\lambda,\mu})$ *if and only if* u_i *is a solution of the equation*

$$\begin{aligned}u_i(t)=\sum_{k=1}^{\ell}a_ku_k(s_k)+\int_0^t\phi_{p_i}^{-1}\Bigg(&x_{u,i,j}-\lambda\int_0^{\delta}F_{u_i}(t,u_1(t),\dots,u_n(t))\mathrm{d}t\\&-\mu\int_0^{\delta}G_{u_i}(t,u_1(t),\dots,u_n(t))\mathrm{d}t\Bigg)\mathrm{d}\delta\end{aligned}$$

for $i=1,\dots,n$, *where* $x_{u,i,j}$ *is the unique solution of* (2.4) *for* $j=0,1,\dots,m$, *and* $\Delta\phi_{p_i}(u_i'(t_j))=I_{ij}(u_i(t_j))$ *for* $i=1,\dots,n$ *and* $j=0,1,\dots,m$.

Proof. This can be verified by direct computations (see [17, Lemma 2.4]). □

Lemma 2.3. *If a function* $u\in E$ *is a weak solution of* $(\phi_{\lambda,\mu})$, *then* u *is a classical solution of* $(\phi_{\lambda,\mu})$.

Proof. Let $u=(u_1,\dots,u_n)\in E$ be a weak solution of $(\phi_{\lambda,\mu})$. Then

$$\begin{aligned}&\int_0^1\left(\sum_{i=1}^n\phi_{p_i}(u_i'(t))v_i'(t)\right)\mathrm{d}t+\sum_{i=1}^n\sum_{j=1}^m I_{ij}(u_i(t_j))v_i(t_j)\\&\quad-\lambda\int_0^1\sum_{i=1}^nF_{u_i}(t,u_1(t),\dots,u_n(t))v_i(t)\mathrm{d}t\\&\quad-\mu\int_0^1\sum_{i=1}^nG_{u_i}(t,u_1(t),\dots,u_n(t))v_i(t)\mathrm{d}t=0\end{aligned} \tag{2.5}$$

for any $v = (v_1, \ldots, v_n) \in \mathrm{W}_0^{1,p_1}([0,1]) \times \mathrm{W}_0^{1,p_2}([0,1]) \times \ldots \times \mathrm{W}_0^{1,p_n}([0,1])$. So (2.5) holds for all $v \in \mathrm{W}_0^{1,p_1}(t_j,t_{j+1}) \times \mathrm{W}_0^{1,p_2}(t_j,t_{j+1}) \times \ldots \times \mathrm{W}_0^{1,p_n}(t_j,t_{j+1})$, $v(t)=0$ for $t \in (0,1) \setminus (t_j,t_{j+1})$ for $j = 0,1,\ldots,m$. Then (2.5) becomes

$$\begin{aligned}
&\int\limits_{t_j}^{t_{j+1}} \left(\sum_{i=1}^{n} \phi_{p_i}(u_i'(t))v_i'(t)\right)\mathrm{d}t - \lambda \int\limits_{t_j}^{t_{j+1}} \sum_{i=1}^{n} F_{u_i}(t,u_1(t),\ldots,u_n(t))v_i(t)\mathrm{d}t \\
&\quad - \mu \int\limits_{t_j}^{t_{j+1}} \sum_{i=1}^{n} G_{u_i}(t,u_1(t),\ldots,u_n(t))v_i(t)\mathrm{d}t = 0
\end{aligned} \tag{2.6}$$

for $j = 0,1,\ldots,m$. Recall that, in one dimension, any weakly differentiable function is absolutely continuous, so that its classical derivative exists almost everywhere, and the classical derivative coincides with the weak derivative. Integrating (2.6) by parts gives

$$\sum_{i=1}^{n} \int\limits_{t_j}^{t_{j+1}} \left[-(\phi_{p_i} \circ u_i')'(t) - \lambda F_{u_i}(t,u_1(t),\ldots,u_n(t)) - \mu G_{u_i}(t,u_1(t),\ldots,u_n(t))\right] v_i(t)\mathrm{d}t = 0$$

for $j = 0,1,\ldots,m$. Thus for $i = 1,\ldots,n$,

$$-(\phi_{p_i} \circ u_i')'(t) - \lambda F_{u_i}(t,u_1(t),\ldots,u_n(t)) - \mu G_{u_i}(t,u_1(t),\ldots,u_n(t)) = 0 \tag{2.7}$$

for almost every $t \in (t_j,t_{j+1})$, $j = 0,1,\ldots,m$. Then, by Lemmas 2.1 and 2.2, we see that

$$\begin{aligned}
u_i(t) = &\sum_{k=1}^{\ell} a_k u_k(s_k) + \int\limits_0^t \phi_{p_i}^{-1}\Bigg(x_{u,i,j} - \lambda \int\limits_0^{\delta} F_{u_i}(t,u_1(t),\ldots,u_n(t))\mathrm{d}t \\
&- \mu \int\limits_0^{\delta} G_{u_i}(t,u_1(t),\ldots,u_n(t))dt\Bigg)\mathrm{d}\delta
\end{aligned}$$

for $i = 1,\ldots,n$, where $x_{u,i,j}$ is the unique solution of (2.4) for $j = 0,1,\ldots,m$. Hence, $u_i \in \mathrm{C}^1(t_j,t_{j+1})$ and $\phi_{p_i} \circ u_i' \in C^1(t_j,t_{j+1})$ for $i = 1,\ldots,n$ and $j = 0,1,\ldots,m$. Therefore (2.7) holds for $t \in (0,1)\backslash Q$. Now we shall show that the impulsive conditions are satisfied. For all $v = (v_1,\ldots,v_n) \in W_0^{1,p_1}([0,1]) \times W_0^{1,p_2}([0,1]) \times \ldots \times W_0^{1,p_n}([0,1])$, from the equality

$$(\phi_{p_i} \circ u_i')'(t)v_i(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int\limits_0^t (\phi_{p_i} \circ u_i')'(\zeta)\mathrm{d}\zeta v_i(t)\right) - \left(\int\limits_0^t (\phi_{p_i} \circ u_i')'(\zeta)\mathrm{d}\zeta\right) v_i'(t),$$

we have

$$\begin{aligned}
&\int_0^1 (\phi_{p_i} \circ u_i')'(t) v_i(t) \mathrm{d}t \\
&= \int_0^1 \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_0^t (\phi_{p_i} \circ u_i')'(\zeta) \mathrm{d}\zeta \, v_i(t) \right) - \left(\int_0^t (\phi_{p_i} \circ u_i')'(\zeta) \mathrm{d}\zeta \right) v_i'(t) \right] dt \\
&= v_i(1) \int_0^1 (\phi_{p_i} \circ u_i')'(t) \mathrm{d}t \\
&\quad - \int_0^1 \left[\phi_{p_i}(u_i'(t)) - \phi_{p_i}(u_i'(0)) - \sum_{0 \le t_j < t} \Delta\phi_{p_i}(u_i'(t_j)) \right] v_i'(t) \mathrm{d}t \\
&= \left[\phi_{p_i}(u_i'(1)) - \phi_{p_i}(u_i'(0)) - \sum_{j=1}^m \Delta\phi_{p_i}(u_i'(t_j)) \right] v_i(1) \\
&\quad - \int_0^1 \phi_{p_i}(u_i'(t)) v_i'(t) \mathrm{d}t + \phi_{p_i}(u_i'(0))[v_i(1) - v_i(0)] \\
&\quad + \sum_{j=1}^m \int_{t_j}^{t_{j+1}} \sum_{0 \le t_j < t} \Delta\phi_{p_i}(u_i'(t_j)) v_i'(t) \mathrm{d}t \\
&= \phi_{p_i}(u_i'(1)) v_i(1) - \phi_{p_i}(u_i'(0)) v_i(0) - \sum_{j=1}^m \Delta\phi_{p_i}(u_i'(t_j)) v_i(1) \\
&\quad - \int_0^1 \phi_{p_i}(u_i'(t)) v_i'(t) \mathrm{d}t - \sum_{j=1}^m \Delta\phi_{p_i}(u_i'(t_j)) v_i(t_j) + \sum_{j=1}^m \Delta\phi_{p_i}(u_i'(t_j)) v_i(1) \\
&= -\sum_{j=1}^m \Delta\phi_{p_i}(u_i'(t_j)) v_i(t_j) - \int_0^1 \phi_{p_i}(u_i'(t)) v_i'(t) \mathrm{d}t
\end{aligned} \tag{2.8}$$

for $i = 1, \ldots, n$. Substituting (2.8) into (2.5), we have

$$\begin{aligned}
&\sum_{i=1}^n \int_0^1 \left(-(\phi_{p_i} \circ u_i')'(t) - \lambda F_{u_i}(t, u_1(t), \ldots, u_n(t)) - \mu G_{u_i}(t, u_1(t), \ldots, u_n(t)) \right) v_i(t) \mathrm{d}t \\
&\quad + \sum_{i=1}^n \sum_{j=1}^m [-\Delta\phi_{p_i}(u'(t_j)) v_i(t_j) + I_{ij}(u_i(t_j)) v_i(t_j)] = 0
\end{aligned}$$

for $i = 1, \dots, n$. Since u satisfies (2.7), we have

$$\Delta\phi_{p_i}(u'(t_j))v_i(t_j) = I_{ij}(u_i(t_j))v_i(t_j), \quad j = 0, 1, \dots, m$$

for $i = 1, \dots, n$. So u is a classical solution of $(\phi_{\lambda,\mu})$. □

Our main tool is Theorem 2.4 which has been obtained by Ricceri ([38, Theorem 2]). It is as follows:

If X is a real Banach space, then denote by $\mathcal{W}_X$ the class of all functionals $\Phi : X \to \mathbb{R}$ possessing the following property: If $\{u_n\}$ is a sequence in X converging weakly to $u \in X$ with $\liminf_{n\to\infty} \Phi(u_n) \leq \Phi(u)$, then $\{u_n\}$ has a subsequence converging strongly to u.

For example, if X is uniformly convex and $g : [0, \infty) \to \mathbb{R}$ is a continuous and strictly increasing function, then, by a classical result, the functional $u \to g(\|u\|)$ belongs to the class $\mathcal{W}_X$.

Theorem 2.4. *Let X be a separable and reflexive real Banach space, let $\Phi : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1-functional, belonging to $\mathcal{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*, and let $J : X \to \mathbb{R}$ be a C^1-functional with compact derivative. Assume that Φ has a strict local minimum u_0 with $\Phi(u_0) = J(u_0) = 0$. Finally, setting*

$$\rho = \max\left\{0, \limsup_{\|u\|\to\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to u_0} \frac{J(u)}{\Phi(u)}\right\},$$

$$\sigma = \sup_{u\in\Phi^{-1}((0,\infty))} \frac{J(u)}{\Phi(u)},$$

assume that $\rho < \sigma$. Then for each compact interval $[c, d] \subset (\frac{1}{\sigma}, \frac{1}{\rho})$ (with the conventions $\frac{1}{0} = \infty, \frac{1}{\infty} = 0$), there exists $R > 0$ with the following property: for every $\lambda \in [c, d]$ and every C^1-functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\gamma > 0$ such that, for each $\mu \in [0, \gamma]$,

$$\Phi'(u) = \lambda J'(u) + \mu\Psi'(u)$$

has at least three solutions in X whose norms are less than R.

We refer the reader to the papers [22, 41] in which Theorem 2.4 was successfully employed to ensure the existence of at least three solutions for boundary value problems.

Now for every $u \in E$, we define

$$\Phi(u) := \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} + \sum_{i=1}^{n}\sum_{j=1}^{m} \int_0^{u_i(t_j)} I_{ij}(\zeta)\mathrm{d}\zeta, \tag{2.9}$$

$$J(u) = \int_0^T F(t, u_1(t), \dots, u_n(t))\mathrm{d}t \tag{2.10}$$

and

$$\Psi(u) = \int_0^T G(t, u_1(t), \dots, u_n(t))\mathrm{d}t. \tag{2.11}$$

Standard arguments show that $\Phi - \mu\Psi - \lambda J$ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in E$ is given by

$$\begin{aligned}
&(\Phi' - \mu\Psi' - \lambda J')(u)(v) \\
&= \int_0^1 \left[\sum_{i=1}^n \phi_{p_i}(u_i'(t))v_i'(t)\right] \mathrm{d}t - \lambda \int_0^1 \sum_{i=1}^n F_{u_i}(t, u_1(t), \dots, u_n(t))v_i(t)\mathrm{d}t \\
&\quad - \mu \int_0^1 \sum_{i=1}^n G_{u_i}(t, u_1(t), \dots, u_n(t))v_i(t)\mathrm{d}t + \sum_{i=1}^n \sum_{j=1}^m I_{ij}(u_i(t_j))v_i(t_j)
\end{aligned}$$

for all $v = (v_1, \dots, v_n) \in \mathrm{W}_0^{1,p_1}([0,1]) \times \mathrm{W}_0^{1,p_2}([0,1]) \times \dots \times \mathrm{W}_0^{1,p_n}([0,1])$. Hence, a critical point of the functional $\Phi - \mu\Psi - \lambda J$ gives us a weak solution of $(\phi_{\lambda,\mu})$, and in view of Lemma 2.3, every weak solution of the problem $(\phi_{\lambda,\mu})$ is a classical one.

We suppose that the impulsive terms satisfy the condition

(I) $I_{ij} : \mathbb{R} \to \mathbb{R}$ is nondecreasing, $I_{ij}(0) = 0$ and $I_{ij}(s)s > 0$, $s \neq 0$ for $i = 1, \dots, n$ and $j = 1, \dots, m$.

We need the following proposition in the proof of our main result.

Proposition 2.5. *Assume that* $(H1)$ *holds. Let* $S : E \to E^*$ *be the operator defined by*

$$S(u)(v) = \int_0^1 \left[\sum_{i=1}^n \phi_{p_i}(u_i'(t))v_i'(t)\right] \mathrm{d}t + \sum_{i=1}^n \sum_{j=1}^m I_{ij}(u_i(t_j))v_i(t_j)$$

for every $u, v \in E$. *Then,* S *admits a continuous inverse on* E^*.

Proof. In the proof, we use C_1, C_2 and C_3 to denote appropriate positive constants. By (I), we have

$$\begin{aligned}
S(u)(u) &= \sum_{i=1}^{n} \int_0^1 \phi_{p_i}(u_i'(t))u_i'(t)\mathrm{d}t + \sum_{i=1}^{n}\sum_{j=1}^{m} I_{ij}(u_i(t_j))u_i(t_j) \\
&= \sum_{i=1}^{n} \int_0^1 |u_i'(t)|^{p_i}\mathrm{d}t + \sum_{i=1}^{n}\sum_{j=1}^{m} I_{ij}(u_i(t_j))u_i(t_j) \\
&\geq \sum_{i=1}^{n} \left[\left(\int_0^1 |u_i'(t)|^{p_i}\mathrm{d}t\right)^{1/p_i}\right]^{p_i} \\
&\geq \sum_{i=1}^{n} \left[\left(\int_0^1 |u_i'(t)|^{p_i}\mathrm{d}t\right)^{1/p_i}\right]^{\underline{p}} \\
&\geq C_1 \left[\sum_{i=1}^{n}\left(\int_0^1 |u_i'(t)|^{p_i}\mathrm{d}t\right)^{1/p_i}\right]^{\underline{p}} = C_1\|u\|^{\underline{p}}.
\end{aligned}$$

This implies that S is coercive. Now, for any $u \in (u_1, \ldots, u_n) \in E$ and $v \in (v_1, \ldots, v_n) \in E$, we have

$$\begin{aligned}
\langle S(u) - S(v), u - v\rangle = &\int_0^1 \left[\sum_{i=1}^{n} \left(\phi_{p_i}(u_i'(t)) - \phi_{p_i}(v_i'(t))\right)\left(u_i'(t) - v_i'(t)\right)\right]\mathrm{d}t \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{m} \left(I_{ij}(u_i(t)) - I_{ij}(v_i(t))\right)\left(u_i(t) - v_i(t)\right).
\end{aligned}$$

Then, by [40, Relation (2.2)] and (I), we see that

$$\langle S(u) - S(v), u - v\rangle \geq \begin{cases} C_2 \sum_{i=1}^{n} \int_0^1 |u_i'(t) - v_i'(t)|^{p_i}\mathrm{d}t, & \underline{p} \geq 2, \\ \\ C_3 \sum_{i=1}^{n} \int_0^1 \frac{|u_i'(t) - v_i'(t)|^2}{(|u_i'(t)| + |v_i'(t)|)^{2-p_i}}\mathrm{d}t, & \overline{p} < 2. \end{cases} \tag{2.12}$$

Now by the same argument as given in the proof of [17, Lemma 2.6] and the condition $(H1)$, if $\underline{p} \geq 2$, we have that S is uniformly monotone. Moreover, since E is reflexive, for $u_n \to u$ strongly in E as $n \to \infty$, one has $S(u_n) \to S(u)$ weakly in E^* as $n \to \infty$. Hence, S is demicontinuous, so by [47, Theorem 26.A(d)], the inverse operator S^{-1} of S exists and it is continuous on E^*. If $\underline{p} < 2$, by (I) and by the same reasoning as in the proof of [17, Lemma 2.6], S is strictly monotone. Thus, by [47, Theorem 26.A(d)],

S^{-1} exists and it is bounded. So by [17, Relation (2.8)], S^{-1} is locally Lipschitz continuous and hence continuous. This completes the proof. □

Remark 2.6. Suppose that the condition (I) is replaced by the condition

(I') $I_{ij}:\mathbb{R}\to\mathbb{R}$ is odd and nondecreasing in $\mathbb{R}$ for any $i=1,\dots,n$ and $j=1,\dots,m$.

Then the conclusion of Proposition 2.5 holds again. It is easy to see that the condition (I') implies the condition (I).

3. MAIN RESULTS

In this section, we formulate our main results. Let us denote by $\mathcal{F}$ the class of all functions $F:[0,1]\times\mathbb{R}^n\to\mathbb{R}$ that are measurable with respect to t, for all $\xi\in\mathbb{R}^n$, continuously differentiable in ξ, for almost every $t\in[0,1]$, satisfy the standard summability condition (1.1). Set

$$A:=\frac{1}{2}\left(1+\frac{\sum_{k=1}^{\ell}|a_k|}{|1-\sum_{k=1}^{\ell}a_k|}+\frac{\sum_{k=1}^{\ell}|b_k|}{|1-\sum_{k=1}^{\ell}b_k|}\right).$$

Let

$$\lambda_1=\inf_{u\in E}\left\{\frac{\sum_{i=1}^n\frac{\|u_i\|_{p_i}^{p_i}}{p_i}+\sum_{i=1}^n\sum_{j=1}^m\int\limits_0^{u_i(t_j)}I_{ij}(\zeta)\mathrm{d}\zeta}{\int\limits_0^1 F(t,u(t))\mathrm{d}t}:\int\limits_0^1 F(t,u(t))\mathrm{d}t>0\right\}$$

and $\lambda_2=\frac{1}{\max\{0,\lambda_0,\lambda_\infty\}}$, where

$$\lambda_0=\limsup_{|u|\to 0}\frac{\int\limits_0^1 F(t,u(t))\mathrm{d}t}{\sum_{i=1}^n\frac{\|u_i\|_{p_i}^{p_i}}{p_i}+\sum_{i=1}^n\sum_{j=1}^m\int\limits_0^{u_i(t_j)}I_{ij}(\zeta)\mathrm{d}\zeta}$$

and

$$\lambda_\infty=\limsup_{\|u\|\to\infty}\frac{\int\limits_0^1 F(t,u(t))\mathrm{d}t}{\sum_{i=1}^n\frac{\|u_i\|_{p_i}^{p_i}}{p_i}+\sum_{i=1}^n\sum_{j=1}^m\int\limits_0^{u_i(t_j)}I_{ij}(\zeta)\mathrm{d}\zeta}$$

with $u=(u_1,\dots,u_n)$.

Theorem 3.1. *Suppose that $F \in \mathcal{F}$. Assume that*

$(\mathcal{A}_1)$ *there exists a constant $\varepsilon > 0$ such that*

$$\max\left\{\limsup_{u\to(0,\dots,0)} \frac{\max_{t\in[0,1]} F(t,u)}{\sum_{i=1}^n |u_i|^{p_i}}, \limsup_{|u|\to\infty} \frac{\max_{t\in[0,1]} F(t,u)}{\sum_{i=1}^n |u_i|^{p_i}}\right\} < \varepsilon,$$

where $u = (u_1, \dots, u_n)$ with $|u| = \sqrt{\sum_{i=1}^n u_i^2}$;

$(\mathcal{A}_2)$ *there exists a function $w \in E$ such that*

$$\sum_{i=1}^n \frac{\|w_i\|_{p_i}^{p_i}}{p_i} + \sum_{i=1}^n \sum_{j=1}^m \int_0^{w_i(t_j)} I_{ij}(\zeta)\mathrm{d}\zeta \neq 0$$

and

$$A\overline{p}\varepsilon < \frac{\int_0^1 F(t, w(t))\mathrm{d}t}{\sum_{i=1}^n \frac{\|w_i\|_{p_i}^{p_i}}{p_i} + \sum_{i=1}^n \sum_{j=1}^m \int_0^{w_i(t_j)} I_{ij}(\zeta)\mathrm{d}\zeta}.$$

Then, for each compact interval $[c, d] \subset (\lambda_1, \lambda_2)$, there exists $R > 0$ with the following property: for every $\lambda \in [c, d]$ and every $G \in \mathcal{F}$, there exists $\gamma > 0$ such that, for each $\mu \in [0, \gamma]$, the system $(\phi_{\lambda,\mu})$ has at least three classical solutions whose norms in E are less than R.

Proof. Take $X = E$. Clearly, X is a separable and uniformly convex Banach space. Let the functionals Φ, J and Ψ be as given in (2.9), (2.10) and (2.11), respectively. The functional Φ is C^1, and due to Proposition 2.5 its derivative admits a continuous inverse on X^*. Moreover, by the sequentially weakly lower semicontinuity of $\sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i}$ and the continuity of I_{ij}, $i = 1, \dots, n$, $j = 1, \dots, m$, Φ is sequentially weakly lower semicontinuous in X. On the other hand, we let $\|u_i\|_{p_i} > 1$, $i = 1, \dots, n$, by (I),

$$\Phi(u) = \sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i} + \sum_{i=1}^n \sum_{j=1}^m \int_0^{u_i(t_j)} I_{ij}(\zeta)\mathrm{d}\zeta \geq \sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i} \geq \frac{1}{\overline{p}}\|u\|^{\underline{p}} \tag{3.1}$$

for every $u = (u_1, \dots, u_n) \in X$. Thus Φ is coercive. Moreover, let M be a bounded subset of X. That is, there exist constants $c_i > 0$, $i = 1, \dots, n$ such that $\|u_i\|_i \leq c_i$ for each $u \in M$. Then, we have

$$|\Phi(u)| \leq \frac{1}{\underline{p}} \sum_{i=1}^n \begin{cases} \|c_i\|^{\underline{p}}, & \text{if } \|u_i\|_i \leq 1, \\ \|c_i\|^{\overline{p}}, & \text{if } \|u_i\|_i > 1. \end{cases}$$

Hence Φ is bounded on each bounded subset of X. Furthermore, $\Phi \in \mathcal{W}_X$. Indeed, let the sequence

$$\{u_k\}_{k=1}^\infty = \{(u_{k1}, \dots, u_{kn})\}_{k=1}^\infty \subset X, \quad u = (u_1, \dots, u_n) \subset X, \quad u_k \rightharpoonup u$$

and $\liminf_{k\to\infty}\Phi(u_k) \leq \Phi(u)$. Since the function I_{ij} is continuous, $i = 1,\dots,n$, $j = 1,\dots,m$, one has

$$\liminf_{n\to\infty}\sum_{i=1}^{n}\frac{\|u_{ki}\|_{p_i}^{p_i}}{p_i} \leq \sum_{i=1}^{n}\frac{\|u_i\|_{p_i}^{p_i}}{p_i}.$$

Thus, $\{u_k\}_{k=1}^{\infty}$ has a subsequence converging strongly to u. Therefore, $\Phi \in \mathcal{W}_X$. The functionals J and Ψ are two C^1-functionals with compact derivatives. Moreover, Φ has a strict local minimum 0 with $\Phi(0) = J(0) = 0$. In view of $(\mathcal{A}_1)$, there exist τ_1, τ_2 with $0 < \tau_1 < \tau_2$ such that

$$F(t,u) \leq \varepsilon\sum_{i=1}^{n}|u_i|^{p_i} \tag{3.2}$$

for every $t \in [0,1]$ and every $u = (u_1,\dots,u_n)$ with $|u| \in [0,\tau_1) \cup (\tau_2,\infty)$. By (1.1), $F(t,u)$ is bounded on $[0,1]\times[\tau_1,\tau_2]$. Thus we can choose $\eta > 0$ and $\upsilon > \overline{p}$ such that

$$F(t,u) \leq \varepsilon\sum_{i=1}^{n}|u_i|^{p_i} + \eta\sum_{i=1}^{n}|u_i|^{\upsilon}$$

for all $(t,u) \in [0,1]\times\mathbb{R}^n$. So, by (2.2), we have

$$J(u) \leq A\varepsilon\sum_{i=1}^{n}\|u_i\|_{p_i}^{p_i} + \eta\sum_{i=1}^{n}A^{\upsilon/p_i}\|u_i\|_{p_i}^{\upsilon} \tag{3.3}$$

for all $u \in X$. Hence, from (3.3), we have

$$\limsup_{|u|\to 0}\frac{J(u)}{\Phi(u)} \leq A\overline{p}\varepsilon. \tag{3.4}$$

Moreover, by using (3.2), for each $u \in X\setminus\{0\}$, we obtain

$$\begin{aligned}
\frac{J(u)}{\Phi(u)} &= \frac{\int\limits_{|u|\leq\tau_2} F(t,u(t))\mathrm{d}t}{\Phi(u)} + \frac{\int\limits_{|u|>\tau_2} F(t,u(t))\mathrm{d}t}{\Phi(u)}\\
&\leq \frac{\sup_{t\in[0,1],|u|\in[0,\tau_2]}F(t,u)}{\Phi(u)} + \frac{A\varepsilon\sum_{i=1}^{n}\|u_i\|_{p_i}^{p_i}}{\Phi(u)}\\
&\leq \frac{\sup_{t\in[0,1],|u|\in[0,\tau_2]}F(t,u)}{\sum_{i=1}^{n}\frac{\|u_i\|_{p_i}^{p_i}}{p_i}} + A\overline{p}\varepsilon.
\end{aligned}$$

So, we get

$$\limsup_{\|u\|\to\infty}\frac{J(u)}{\Phi(u)} \leq A\overline{p}\varepsilon. \tag{3.5}$$

In view of (3.4) and (3.5), we have

$$\rho = \max\left\{0,\ \limsup_{\|u\|\to\infty}\frac{J(u)}{\Phi(u)},\ \limsup_{u\to(0,\dots,0)}\frac{J(u)}{\Phi(u)}\right\} \leq A\overline{p}\varepsilon. \tag{3.6}$$

Assumption $(\mathcal{A}_2)$ in conjunction with (3.6) yields

$$\sigma = \sup_{u\in\Phi^{-1}((0,\infty))} \frac{J(u)}{\Phi(u)} = \sup_{X\setminus\{0\}} \frac{J(u)}{\Phi(u)}$$
$$\geq \frac{\int_0^1 F(t,w(t))\mathrm{d}t}{\Phi(w(t))} = \frac{\int_0^1 F(t,w(t))\mathrm{d}t}{\sum_{i=1}^n \frac{\|w_i\|_{p_i}^{p_i}}{p_i} + \sum_{i=1}^n\sum_{j=1}^m \int_0^{w_i(t_j)} I_{ij}(\zeta)\mathrm{d}\zeta} > A\overline{p}\varepsilon \geq \rho.$$

Thus, all the hypotheses of Theorem 2.4 are satisfied. Clearly, $\lambda_1 = \frac{1}{\beta}$ and $\lambda_2 = \frac{1}{\alpha}$. Then, using Theorem 2.4, taking Lemma 2.3 into account, for each compact interval $[c,d] \subset (\lambda_1,\lambda_2)$, there exists $R > 0$ with the following property: for every $\lambda \in [c,d]$ and every $G \in \mathcal{F}$ there exists $\gamma > 0$ such that, for each $\mu \in [0,\gamma]$, the system $(\phi_{\lambda,\mu})$ has at least three classical solutions whose norms in E are less than R. □

Another announced application of Theorem 2.4 reads as follows.

Theorem 3.2. *Suppose that $F \in \mathcal{F}$. Assume that*

$$\max\left\{\limsup_{u\to(0,\dots,0)} \frac{\max_{t\in[0,1]} F(t,u)}{\sum_{i=1}^n |u_i|^{p_i}}, \limsup_{|u|\to\infty} \frac{\max_{t\in[0,1]} F(t,u)}{\sum_{i=1}^n |u_i|^{p_i}}\right\} < 0 \tag{3.7}$$

where $u = (u_1,\dots,u_n)$ with $|u| = \sqrt{\sum_{i=1}^n u_i^2}$, and

$$\sup_{u\in E} \frac{\int_0^1 F(t,u(t))\mathrm{d}t}{\sum_{i=1}^n \frac{\|u_i\|_{p_i}^{p_i}}{p_i} + \sum_{i=1}^n\sum_{j=1}^m \int_0^{u_i(t_j)} I_{ij}(\zeta)\mathrm{d}\zeta} > 0. \tag{3.8}$$

Then for each compact interval $[c,d] \subset (\lambda_1,\infty)$ there exists $R > 0$ with the following property: for every $\lambda \in [c,d]$ and every $G \in \mathcal{F}$, there exists $\gamma > 0$ such that for each $\mu \in [0,\gamma]$, the system $(\phi_{\lambda,\mu})$ has at least three classical solutions whose norms in E are less than R.

Proof. In view of (3.7), there exist an arbitrary $\varepsilon > 0$ and τ_1, τ_2 with $0 < \tau_1 < \tau_2$ such that

$$F(t,u) \leq \varepsilon \sum_{i=1}^n |u_i|^{p_i}$$

for every $t \in [0,1]$ and every $u = (u_1,\dots,u_n)$ with $|u| \in [0,\tau_1) \cup (\tau_2,\infty)$. By (1.1), $F(t,u)$ is bounded on $[0,1]\times[\tau_1,\tau_2]$. Thus we can choose $\eta > 0$ and $\upsilon > \overline{p}$ in a manner that

$$F(t,u) \leq \varepsilon \sum_{i=1}^n |u_i|^{p_i} + \eta \sum_{i=1}^n |u_i|^{\upsilon}$$

for all $(t,u) \in [0,1] \times \mathbb{R}^n$. So, by the same process as in the proof of Theorem 3.1, we obtain (3.4) and (3.5). Since ϵ is arbitrary, (3.4) and (3.5) give

$$\max\left\{0, \limsup_{\|u\|\to+\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to(0,\dots,0)} \frac{J(u)}{\Phi(u)}\right\} \le 0.$$

Then, with the notation of Theorem 2.4, we have $\rho = 0$. By (3.8), we also have $\sigma > 0$. In this case, clearly $\lambda_1 = \frac{1}{\sigma}$ and $\lambda_2 = \infty$. Thus, by using Theorem 2.4 and taking Lemma 2.3 into account, the result is achieved. □

Remark 3.3. Theorem 1.1 immediately follows from Theorem 3.2.

Remark 3.4. In Assumption $(\mathcal{A}_2)$, if we choose

$$w(t) = w^\star(t) = (0,\dots,0,w_n(t)) \tag{3.9}$$

with

$$w_n(t) = \begin{cases} \delta\left(\sum_{k=1}^{\ell} a_k + \frac{2\left(1-\sum_{k=1}^{\ell} a_k\right)}{s_1} t\right), & \text{if } t \in [0, \frac{s_1}{2}), \\ \delta, & \text{if } t \in [\frac{s_1}{2}, \frac{1+s_\ell}{2}], \\ \delta\left(\frac{2-\sum_{k=1}^{\ell} b_k - s_\ell \sum_{k=1}^{\ell} b_k}{1-s_\ell} - \frac{2(1-\sum_{k=1}^{\ell} b_k)}{1-s_\ell} t\right), & \text{if } t \in (\frac{1+s_\ell}{2}, 1], \end{cases} \tag{3.10}$$

where $\delta > 0$, then $(\mathcal{A}_2)$ takes the following form:

$(\mathcal{A}_2')$ there exists a positive constant δ such that

$$(\omega_n\delta)^{p_n} + p_n \sum_{j=1}^{m} \int_0^{w_n(t_j)} I_{nj}(\zeta)\mathrm{d}\zeta \neq 0$$

and

$$A\overline{p}\varepsilon < \frac{p_n \int_0^1 F(t, w^\star(t))\mathrm{d}t}{(\omega_n\delta)^{p_n} + p_n \sum_{j=1}^{m} \int_0^{w_n(t_j)} I_{nj}(\zeta)\mathrm{d}\zeta}.$$

Clearly, $w^\star \in E$ and

$$\Phi(w^\star) = \frac{(\omega_n\delta)^{p_n}}{p_n} + \sum_{j=1}^{m} \int_0^{w_n(t_j)} I_{nj}(\zeta)\mathrm{d}\zeta,$$

where

$$\omega_n := \left[2^{p_n-1}\left(s_1^{1-p_n}\left|1-\sum_{k=1}^{\ell} a_k\right|^{p_n} + (1-s_\ell)^{1-p_n}\right)\left|1-\sum_{k=1}^{\ell} b_k\right|^{p_n}\right]^{1/p_n}.$$

Now, we point out some results in which the function F has separated variables. To be precise, we consider the system

$$\begin{cases} -(\phi_{p_i}(u_i'))' = \lambda\theta(t)F_{u_i}(u_1,\dots,u_n) + \mu G_{u_i}(t,u_1,\dots,u_n), & t\in(0,1)\setminus Q,\\ \Delta\phi_{p_i}(u_i'(t_j)) = I_{ij}(u_i(t_j)), \quad j=1,\dots,m, \\ u_i(0)=\sum_{k=1}^{\ell} a_k u_i(s_k), \quad u_i(1)=\sum_{k=1}^{\ell} b_k u_i(s_k), \end{cases} \tag{$\phi^{\theta}_{\lambda,\mu}$}$$

where $\theta:[0,1]\to\mathbb{R}$ is a nonzero function such that $\theta\in \mathrm{L}^1([0,1])$ and $F:\mathbb{R}^n\to\mathbb{R}$ is a C^1-function and $G:[0,1]\times\mathbb{R}^n\to\mathbb{R}$ is as introduced for the system $(\phi_{\lambda,\mu})$ in Section 1.

Set $F(t,x_1,\dots,x_n)=\theta(t)F(x_1,\dots,x_n)$ for every $(t,x_1,\dots,x_n)\in[0,1]\times\mathbb{R}^n$. The following existence results are consequences of Theorem 3.1.

Theorem 3.5. *Assume that*

($\mathcal{A}_1'$) *there exists a constant $\varepsilon>0$ such that*

$$\sup_{t\in[0,1]}\theta(t).\max\left\{\limsup_{u\to(0,\dots,0)}\frac{F(u)}{\sum_{i=1}^n|u_i|^{p_i}},\ \limsup_{|u|\to\infty}\frac{F(u)}{\sum_{i=1}^n|u_i|^{p_i}}\right\}<\varepsilon,$$

where $u=(u_1,\dots,u_n)$ with $|u|=\sqrt{\sum_{i=1}^n u_i^2}$;

($\mathcal{A}_2''$) *there exists a positive constant δ such that*

$$(\omega_n\delta)^{p_n}+p_n\sum_{j=1}^m\int_0^{w_n(t_j)} I_{nj}(\zeta)\mathrm{d}\zeta\neq 0$$

and

$$A\overline{p}\varepsilon<\frac{p_n\int_0^1 F(t,w^\star(t))\mathrm{d}t}{(\omega_n\delta)^{p_n}+p_n\sum_{j=1}^m\int_0^{w_n(t_j)} I_{nj}(\zeta)\mathrm{d}\zeta}$$

where $w^\star$ and ω_n are given by (3.9) *and* (3.10)*, respectively.*

Then, for each compact interval $[c,d]\subset(\lambda_3,\lambda_4)$, where λ_3 and λ_4 are the same as λ_1 and λ_2, but $\int_0^1 F(t,u(t))\mathrm{d}t$ is replaced by $\int_0^1\theta(t)F(u(t))\mathrm{d}t$, respectively, there exists $R>0$ with the following property: for every $\lambda\in[c,d]$ and every $G\in\mathcal{F}$, there exists $\gamma>0$ such that for each $\mu\in[0,\gamma]$, the system $(\phi^{\theta}_{\lambda,\mu})$ has at least three classical solutions whose norms in E are less than R.

Theorem 3.6. *Assume that there exists a positive constant δ such that*

$$(\omega_n\delta)^{p_n}+p_n\sum_{j=1}^m\int_0^{w_n(t_j)} I_{nj}(\zeta)\mathrm{d}\zeta>0 \ \text{and}\ \int_0^1\theta(t)F(w^\star(t))\mathrm{d}t>0, \tag{3.11}$$

where $w^\star$ *and* ω_n *are given by* (3.9) *and* (3.10)*, respectively. Moreover, suppose that*

$$\limsup_{u\to(0,\ldots,0)} \frac{F(u)}{\sum_{i=1}^n |u_i|^{p_i}} = \limsup_{|u|\to\infty} \frac{F(u)}{\sum_{i=1}^n |u_i|^{p_i}} = 0, \tag{3.12}$$

where $u=(u_1,\ldots,u_n)$ *with* $|u|=\sqrt{\sum_{i=1}^n u_i^2}$*. Then, for each compact interval* $[c,d]\subset(\lambda_3,\infty)$ *where* λ_3 *is the same as* λ_1 *but* $\int_0^1 F(t,u(t))\mathrm{d}t$ *is replaced by* $\int_0^1 \theta(t)F(u(t))\mathrm{d}t$*, there exists* $R>0$ *with the following property: for every* $\lambda\in[c,d]$ *and every* $G\in\mathcal{F}$*, there exists* $\gamma>0$ *such that for each* $\mu\in[0,\gamma]$*, the system* $(\phi^\theta_{\lambda,\mu})$ *has at least three classical solutions whose norms in* E *are less than* R*.*

Proof. We easily observe that, from (3.12), the assumption $(\mathcal{A}_1')$ is satisfied for every $\varepsilon>0$. Moreover, using (3.11), by choosing $\varepsilon>0$ small enough, one can derive the assumption $(\mathcal{A}_2'')$. Hence, the conclusion follows from Theorem 3.5. $\square$

Now, we exhibit an example in which the hypotheses of Theorem 3.6 are satisfied.

Example 3.7. Let $n=2$, $p_1=p_2=2$, $m=l=1$, $t_1=\frac{1}{3}$, $s_1=\frac{1}{2}$, $a_1=b_1=2$ and $I_{11}(x)=I_{21}(x)=x^3$ for each $x\in\mathbb{R}$. Let $\theta(t)=[t]+1$ for all $t\in[0,1]$, where $[t]$ is the integer part of t and

$$F(x_1,x_2)=\begin{cases}(x_1^2+x_2^2)^2, & \text{if } x_1^2+x_2^2<1,\\ 1, & \text{if } x_1^2+x_2^2\ge 1.\end{cases}$$

By choosing $\delta=1$, we have $w(t)=w^\star(t)=(0,w_2(t))$ with

$$w_2(t)=\begin{cases}2(1-2t) & \text{if } t\in[0,\frac{1}{4}),\\ 1 & \text{if } t\in[\frac{1}{4},\frac{3}{4}],\\ -2(1-2t) & \text{if } t\in(\frac{3}{4},1],\end{cases}$$

and $\omega_2=2\sqrt{2}$. Thus we have

$$(\omega_2\delta)^{p_2}+p_2\sum_{j=1}^m \int_0^{w_2(t_j)} I_{2j}(\zeta)\mathrm{d}\zeta=(2\sqrt{2})^2+2\int_0^1\zeta^3\mathrm{d}\zeta=\frac{17}{2}>0,$$

$$\begin{aligned}\int_0^1 \theta(t)F(w^\star(t))\mathrm{d}t &= \int_0^{\frac{1}{4}} F(0,2(1-2t))\mathrm{d}t+\int_{\frac{1}{4}}^{\frac{3}{4}} F(0,1)\mathrm{d}t+\int_{\frac{3}{4}}^1 F(0,-2(1-2t))\mathrm{d}t\\ &= \int_0^{\frac{1}{4}} 16(1-2t)^4\mathrm{d}t+\int_{\frac{1}{4}}^{\frac{3}{4}}\mathrm{d}t+\int_{\frac{3}{4}}^1\mathrm{d}t=\frac{23}{10}>0,\end{aligned}$$

$$\lim_{(u_1,u_2)\to(0,0)}\frac{F(u_1,u_2)}{u_1^2+u_2^2}=\lim_{(u_1,u_2)\to(0,0)}\frac{(u_1^2+u_2^2)^2}{u_1^2+u_2^2}=\lim_{(u_1,u_2)\to(0,0)}(u_1^2+u_2^2)=0,$$

and

$$\lim_{|u|\to\infty} \frac{F(u_1,u_2)}{u_1^2+u_2^2} = \lim_{|u|\to\infty} \frac{1}{u_1^2+u_2^2} = 0,$$

where $u=(u_1,u_2)$ with $|u|=\sqrt{u_1^2+u_2^2}$. Hence, by applying Theorem 3.6 for each compact interval $[c,d]\subset(0,\infty)$, there exists $R>0$ with the following property: for every $\lambda\in[c,d]$ and every $G\in\mathcal{F}$, there exists $\gamma>0$ such that, for each $\mu\in[0,\gamma]$, the system

$$\begin{cases} -u_1''(t) = \lambda([t]+1)F_{u_1}(u_1,u_2) + \mu G_{u_1}(t,u_1,u_2), & t\in(0,1)\setminus\{\frac{1}{3}\},\\ -u_2''(t) = \lambda([t]+1)F_{u_2}(u_1,u_2) + \mu G_{u_2}(t,u_1,u_2), & t\in(0,1)\setminus\{\frac{1}{3}\},\\ u_1'(\frac{1}{3}^{+}) - u_1'(\frac{1}{3}^{-}) = (u_1(\frac{1}{3}))^3, \\ u_2'(\frac{1}{3}^{+}) - u_2'(\frac{1}{3}^{-}) = (u_2(\frac{1}{3}))^3, \\ u_1(0)=2u_1(\frac{1}{2}), \quad u_1(1)=2u_1(\frac{1}{2}), \\ u_2(0)=2u_2(\frac{1}{2}), \quad u_2(1)=2u_2(\frac{1}{2}) \end{cases}$$

has at least three classical solutions whose norms in the space

$$\widetilde{E} = \Big\{(\xi_1,\xi_2)\in \mathrm{W}^{1,2}([0,1])\times \mathrm{W}^{1,2}([0,1]) : \xi_1(0)=\xi_1(1)=2\xi_1\left(\frac{1}{2}\right), \\ \xi_2(0)=\xi_2(1)=2\xi_2\left(\frac{1}{2}\right)\Big\}$$

are less than R.

4. SINGLE IMPULSE

As an application of the results from Section 3, we consider the problem

$$\begin{cases} -(\phi_p(u'))' = \lambda f(t,u) + \mu g(t,u), & t\in(0,1)\setminus\{t_1\},\\ \Delta\phi_p(u'(t_1)) = I(u(t_1)), \\ u(0)=u(1)=u(s_1), \end{cases} \tag{4.1}$$

where $p>1$, $0<t_1,s_1<1$, $t_1\neq s_1$, $\phi_p(x)=|x|^{p-2}x$, $\lambda>0, \mu\geq 0$, and $f,g:[0,1]\times\mathbb{R}\to\mathbb{R}$ are two L^1-Carathéodory functions, $\Delta\phi_p u_i'(t_1)=\phi_p u'(t_1^+)-\phi_p u'(t_1^-)$, where $u'(t_1^+)$ and $u'(t_1^-)$ represent the right-hand limit and left-hand limit of $u'(t)$ at $t=t_1$, respectively, and $I:\mathbb{R}\to\mathbb{R}$ is a continuous and nondecreasing function such that $I(0)=0$ and $I(s)s>0$ for all $s\in\mathbb{R}\setminus\{0\}$.

We introduce the functions $F:[0,1]\times\mathbb{R}\to\mathbb{R}$ and $G:[0,1]\times\mathbb{R}\to\mathbb{R}$, respectively, as

$$F(t,x) = \int_0^x f(t,\zeta)\mathrm{d}\zeta \quad \text{for all } (t,x)\in[0,1]\times\mathbb{R}$$

and

$$G(t,x)=\int_0^x g(t,\zeta)\mathrm{d}\zeta \quad \text{for all } (t,x)\in[0,1]\times\mathbb{R}.$$

The following two theorems are consequences of Theorems 3.1 and 3.2, respectively.

Theorem 4.1. *Assume that*

($\mathcal{B}_1$) *there exists a constant $\varepsilon>0$ such that*

$$\max\left\{\limsup_{u\to 0}\frac{\max_{t\in[0,1]}F(t,u)}{|u|^p},\ \limsup_{|u|\to\infty}\frac{\max_{t\in[0,1]}F(t,u)}{|u|^p}\right\}<\varepsilon;$$

($\mathcal{B}_2$) *there exists a function $w\in\widehat{E}$, where*

$$\widehat{E}=\left\{\xi\in \mathrm{W}^{1,p}([0,1])\ :\ \xi(0)=\xi(1)=\xi(s_1)\right\},$$

such that

$$\|w\|_p^p+p\int_0^{w(t_1)} I(\zeta)\mathrm{d}\zeta\neq 0$$

and

$$p^2\varepsilon<\frac{2\int_0^1 F(t,w(t))\mathrm{d}t}{\|u\|_p^p+p\int_0^{u(t_1)} I(\zeta)\mathrm{d}\zeta}.$$

Then, for each compact interval $[c,d]\subset(\bar\lambda_1,\bar\lambda_2)$, where

$$\bar\lambda_1=\inf_{u\in\widehat{E}}\left\{\frac{\|u\|_p^p+p\int_0^{u(t_1)} I(\zeta)\mathrm{d}\zeta}{p\int_0^1 F(t,u(t))\mathrm{d}t}\ :\ \int_0^1 F(t,u(t))\mathrm{d}t>0\right\}$$

and

$$\bar\lambda_2=\max\left\{0,\ \limsup_{|u|\to 0}\frac{p\int_0^1 F(t,u(t))\mathrm{d}t}{\|u\|_p^p+p\int_0^{u(t_1)} I(\zeta)\mathrm{d}\zeta},\ \limsup_{\|u\|_p\to\infty}\frac{p\int_0^1 F(t,u(t))\mathrm{d}t}{\|u\|_p^p+p\int_0^{u(t_1)} I(\zeta)\mathrm{d}\zeta}\right\},$$

there exists $R>0$ with the following property: for every $\lambda\in[c,d]$, there exists $\gamma>0$ such that for each $\mu\in[0,\gamma]$, the problem (4.1) has at least three classical solutions whose norms in $\widehat{E}$ are less than R.

Theorem 4.2. *Assume that*

$$\max\left\{\limsup_{u\to 0}\frac{\sup_{t\in[0,1]}F(t,u(t))}{|u|^p},\ \limsup_{|u|\to\infty}\frac{\sup_{t\in[0,1]}F(t,u(t))}{|u|^p}\right\}\le 0$$

and

$$\sup_{u\in\widehat{E}}\frac{p\int\limits_0^1 F(t,u(t))\mathrm{d}t}{\|u\|_p^p+p\int\limits_0^{u(t_1)} I(\zeta)\mathrm{d}\zeta}>0.$$

Then for each compact interval $[c,d]\subset(\bar{\lambda}_1,\infty)$, where $\bar{\lambda}_1$ is defined in Theorem 4.1, there exists $R>0$ with the following property: for every $\lambda\in[c,d]$, there exists $\gamma>0$ such that for each $\mu\in[0,\gamma]$, the problem (4.1) *has at least three classical solutions whose norms in $\widehat{E}$ are less than R.*

Now let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. Put

$$F(x)=\int\limits_0^x f(\zeta)\mathrm{d}\zeta \text{ for all } x\in\mathbb{R}.$$

The following theorems are immediate consequences of Theorems 3.5 and 3.6, respectively.

Theorem 4.3. *Assume that*

($\mathcal{B}_1'$) *there exists a constant $\varepsilon>0$ such that*

$$\max\left\{\limsup_{u\to 0}\frac{F(u)}{|u|^p},\ \limsup_{|u|\to\infty}\frac{F(u)}{|u|^p}\right\}<\varepsilon;$$

($\mathcal{B}_2''$) *there exists a positive constant δ such that*

$$2^{p(p-1)}\delta^p\left(s_1^{1-p}+(1-s_1)^{1-p}\right)+p\int\limits_0^{\delta} I(\zeta)\mathrm{d}\zeta\neq 0$$

and

$$p\varepsilon<\frac{2p_n\int\limits_0^1 F(\overline{w}(t))\mathrm{d}t}{2^{p(p-1)}\delta^p\left(s_1^{1-p}+(1-s_1)^{1-p}\right)+p\int\limits_0^{\delta} I(\zeta)\mathrm{d}\zeta},$$

where

$$\overline{w}(t)=\begin{cases}\frac{2\delta}{s_1}t, & \text{if } t\in[0,\frac{s_1}{2}),\\ \delta, & \text{if } t\in[\frac{s_1}{2},\frac{1+s_1}{2}],\\ \frac{2\delta}{1-s_1}(1-t), & \text{if } t\in(\frac{1+s_1}{2},1].\end{cases}\tag{4.2}$$

Then, for each compact interval $[c,d] \subset (\bar\lambda_3, \bar\lambda_4)$*, where* $\bar\lambda_3$ *and* $\bar\lambda_4$ *are the same as* $\bar\lambda_1$ *and* $\bar\lambda_2$*, but* $\int_0^1 F(t,u(t))\mathrm{d}t$ *is replaced by* $\int_0^1 F(u(t))\mathrm{d}t$*, respectively, there exists* $R>0$ *with the following property: for every* $\lambda \in [c,d]$ *and every continuous function* $g:\mathbb{R}\to\mathbb{R}$*, there exists* $\gamma>0$ *such that for each* $\mu\in[0,\gamma]$*, the problem*

$$\begin{cases} -(\phi_p(u'))' = \lambda f(u) + \mu g(u), & t \in (0,1)\setminus\{t_1\}, \\ \Delta\phi_p(u'(t_1)) = I(u(t_1)), \\ u(0) = u(1) = u(s_1) \end{cases} \tag{4.3}$$

has at least three classical solutions whose norms in $\widehat{E}$ *are less than* R*.*

Theorem 4.4. *Assume that there exists a positive constant* δ *such that*

$$2^{p(p-1)}\delta^p\left(s_1^{1-p} + (1-s_1)^{1-p}\right) + p\int\limits_0^\delta I(\zeta)\mathrm{d}\zeta > 0 \text{ and } \int\limits_0^1 F(\overline{w}(t))\mathrm{d}t > 0,$$

where $\overline{w}$ *is given by* (4.2)*. Moreover, suppose that*

$$\limsup_{u\to 0}\frac{f(u)}{|u|^{p-1}} = \limsup_{|u|\to\infty}\frac{f(u)}{|u|^{p-1}} = 0.$$

Then, for each compact interval $[c,d]\subset(\bar\lambda_3,+\infty)$*, where* $\bar\lambda_3$ *is the same as* $\bar\lambda_1$*, but* $\int_0^1 F(t,u(t))\mathrm{d}t$ *is replaced by* $\int_0^1 F(u(t))\mathrm{d}t$*, there exists* $R>0$ *with the following property: for every* $\lambda\in[c,d]$ *and every continuous function* $g:\mathbb{R}\to\mathbb{R}$*, there exists* $\gamma>0$ *such that for each* $\mu\in[0,\gamma]$*, the problem* (4.3) *has at least three classical solutions whose norms in* $\widehat{E}$ *are less than* R*.*

Finally, we present the following example in order to illustrate Corollary 4.4.

Example 4.5. Let $p=3$, $t_1=\frac{2}{3}$, $s_1=\frac{1}{2}$, $I(x)=\sqrt[3]{x}$ for all $x\in\mathbb{R}$ and $f(x)=-x^3e^{-x}$ for all $x\in\mathbb{R}$. By choosing $\delta=1$, we have

$$\overline{w}(t) = \begin{cases} 4t, & \text{if } t\in[0,\frac{1}{4}), \\ 1, & \text{if } t\in[\frac{1}{4},\frac{3}{4}], \\ 4(1-t), & \text{if } t\in(\frac{3}{4},1]. \end{cases}$$

Thus

$$2^{p(p-1)}\delta^p\left(s_1^{1-p}+(1-s_1)^{1-p}\right) + p\int\limits_0^\delta I(\zeta)\mathrm{d}\zeta = 2^9 + 3\int\limits_0^1 \sqrt[3]{\zeta}\mathrm{d}\zeta = 2^9 + \frac{3}{4} > 0,$$

$$
\begin{aligned}
\int_0^1 F(\overline{w}(t))\mathrm{d}t &= -\int_0^{\frac{1}{4}}\int_0^{4t}\zeta^3e^{-\zeta}\mathrm{d}\zeta\mathrm{d}t - \int_{\frac{1}{4}}^{\frac{3}{4}}\int_0^{1}\zeta^3e^{-\zeta}\mathrm{d}\zeta\mathrm{d}t - \int_{\frac{3}{4}}^{1}\int_0^{4(1-t)}\zeta^3e^{-\zeta}\mathrm{d}\zeta\mathrm{d}t \\
&= \int_0^{\frac{1}{4}}\left[e^{-4t}(64t^3+58t^2+24t+6)-6\right]\mathrm{d}t + \left(\frac{16}{e}-6\right)\int_{\frac{1}{4}}^{\frac{3}{4}}\mathrm{d}t \\
&\quad + \frac{2}{e^4}\int_{\frac{3}{4}}^{1}\left[e^{4t}(76-166t+125t^2-32t^3)-6\right]\mathrm{d}t \\
&= \frac{29}{6}+\frac{8}{e}-\frac{1089}{16e^4}+\frac{122}{64e^7}-\frac{70}{32e^8} > 0.
\end{aligned}
$$

Also $\lim_{u\to 0}\frac{f(u)}{|u|^2} = \lim_{|u|\to\infty}\frac{f(u)}{|u|^2} = 0$. Hence, by applying Corollary 4.4 for each compact interval $[c,d]\subset(0,\infty)$, there exists $R>0$ with the following property: for every $\lambda\in[c,d]$ and every continuous function $g:\mathbb{R}\to\mathbb{R}$, there exists $\gamma>0$ such that, for each $\mu\in[0,\gamma]$, the problem

$$
\begin{cases}
-|u_1'(t)|u_1''(t) = -\lambda u(t)^3e^{-u(t)} + \mu g(u), \quad t\in(0,1)\setminus\{\frac{2}{3}\},\\
|u'(\frac{2}{3}^+)|u'(\frac{2}{3}^+) - |u'(\frac{2}{3}^-)|u_1'(\frac{2}{3}^-) = \sqrt[3]{u(\frac{2}{3})},\\
u(0)=u(1)=u(\frac{1}{2})
\end{cases}
$$

has at least three classical solutions whose norms in the space

$$
\widetilde{E}_1 = \left\{\xi\in \mathrm{W}^{1,3}([0,1]) \,:\, \xi(0)=\xi(1)=\xi\left(\frac{1}{2}\right)\right\}
$$

are less than R.

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