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Fractional-Order Backward-Difference Grünwald–Letnikov and Horner Simplified Forms Evaluation Accuracy Analysis

1. Introduction

The aim of this paper is to investigate two equivalent simplified forms of the fractional-order backward difference (FOBD) [4]. The first form is known as the Grünvald–Letnikow form the second one as the Horner form [1, 3, 4, 5, 6]. The simplifications are forced by microprocessor systems requirements. Two simplified forms are analysed. The investigations are illustrated by numerical examples of simplified FOBD evaluation results.

1.1. Grünvald–Letnikov FOBD form

Definition 1. The Grünvald–Letnikov FOBD of a discrete-time real, bounded function y_k is defined as a following sum

$$ {}_{0}^{GL}\Delta_k^{(\nu)} y_k = \sum_{i=0}^{k} a_i^{(\nu)} y_{k-i} \tag{1} $$

where:
- $v \in \mathbf{R}$ – the FOBD order,
- $y_k \in \mathbf{R}$ – discrete-time function ($y_k = 0$ for $k < 0$),
- k_0, k – the FOBD evaluation range,
- $a_i^{(\nu)}$ – coefficients, defined for $i = 0, 1 ..., k–1, k$.

$$ a_i^{(\nu)} = \begin{cases} 1 & \text{for} \quad i = 0 \\ (-1)^i \dfrac{n(n-1)\cdots(n-i+1)}{i!} & \text{for} \quad i = 1,2,3,\cdots \end{cases} \tag{2} $$

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One can immediately verify that

$$a_i^{(\nu)} = a_{i-1}^{(\nu)}\left(1-\frac{1+\nu}{i}\right), \quad \text{for} \quad i=1,2,\cdots \tag{3}$$

One can easily realised that for

$$y_k = \delta_k \tag{4}$$

where δ_k is the discrete Dirac pulse defined as

$$\delta_k = \begin{cases} 1 & \text{for} \quad k=0 \\ 0 & \text{for} \quad k \neq 0 \end{cases} \tag{5}$$

Then

$${}_{0}^{GL}\Delta_k^{(\nu)}\delta_k = \sum_{i=0}^{k} a_i^{(\nu)}\delta_{k-i} = a_k^{(\nu)} \tag{6}$$

and the definition formula (1) can be expressed in a form

$${}_{0}^{GL}\Delta_k^{(\nu)} y_k = \sum_{i=0}^{k}\left({}_{0}\Delta_k^{(\nu)}\delta_k\right) y_{k-i} \tag{7}$$

which describes a classical discrete convolution formula. Hence one can write

$${}_{0}^{GL}\Delta_k^{(\nu)} y_k = {}_{0}\Delta_k^{(\nu)}\delta_k * y_k = b_k^{(\nu)} * y_k \tag{8}$$

where the asterisk denotes a discrete convolution. Formula (1) can be also expressed as a product of two vectors

$${}_{0}^{GL}\Delta_k^{(\nu)} y_k = \sum_{i=0}^{k} a_i^{(\nu)} y_{k-i} = \begin{bmatrix} 1 & a_1^{(\nu)} & \cdots & a_k^{(\nu)} \end{bmatrix} \begin{bmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_0 \end{bmatrix} \tag{9}$$

Formulae (1) and (8) show that to evaluate the FOBD at k – discrete-time instant one should perform k – multiplications and k – additions.

1.2. Horner FOBD form

Definition 2. The Horner form of the FOBD may be expressed as

$${}_{0}^{H}\Delta_k^{(\nu)} y_k =$$

$$= c_0^{(\nu)}\left[y_k + c_1^{(\nu)}\left[y_{k-1} + c_2^{(\nu)}\left[y_{k-2} + \cdots + c_{k-2}^{(\nu)}\left[y_2 + c_{k-1}^{(\nu)}\left[y_1 + c_k^{(\nu)} y_0\right]\right] \cdots \right]\right]\right] \tag{10}$$

with coefficients

$$c_i^{(\nu)} = \begin{cases} 1 & \text{for} \quad i = 0 \\ \dfrac{i-1-\nu}{i} & \text{for} \quad i = 1,2,3,\cdots \end{cases} \tag{11}$$

One can easily prove that

$${}_0^H\Delta_k^{(\nu)} y_k = {}_0^{GL}\Delta_k^{(\nu)} y_k \tag{12}$$

Immediate calculations show that analogously to (2) for coefficients (11) one can also find a recurrent formula i = 1, 2, 3, ...

$$c_i^{(\nu)} = \frac{1+\nu c_{i-1}^{(\nu)}}{2+\nu - c_{i-1}^{(\nu)}} \tag{13}$$

Denoting

$$\begin{aligned} d_0^{(\nu)} &= y_0 \\ d_i^{(\nu)} &= y_i + c_{k-i+1}^{(\nu)} y_{i-1} \text{ for } i = 1,2,\cdots,k-1 \end{aligned} \tag{14}$$

the Horner form of the FOBD equals simply to

$${}_0^H\Delta_k^{(\nu)} y_k = d_k^{(\nu)} \tag{15}$$

2. “Calculation tail” problem

In practical (real-time) calculations one easily realise that in consecutive steps of the FOBD evaluations number of multiplications and additions increases linearly. This situation finally leads to a lack of time inside a sampling period in witch the FOBD must be calculated and a shortage of available memory. The last mentioned discomfort is also called “a finite memory problem”[6]. An analysis of the coefficients $a_i^{(\nu)}$ and $c_i^{(\nu)}$ values may be helpful in an invention of some remedies against formulated above problems.

2.1. Coefficients $a_i^{(\nu)}$ and $c_i^{(\nu)}$ properties

Now one assumes that

$$0 < \nu < 1 \tag{16}$$

Then

$$\lim_{i\to\infty} a_i^{(\nu)} = 0 \tag{17}$$

and

$$a_i^{(\nu)} < a_{i+1}^{(\nu)} < 0 \quad \text{for} \quad i = 1, 2, \cdots \tag{18}$$

Realising that

$$a_0^{(\nu)} = 1 \tag{19}$$

one can prove that

$$\sum_{i=0}^{\infty} a_i^{(\nu)} = a_0^{(\nu)} + \sum_{i=1}^{\infty} a_i^{(\nu)} = 1 + \sum_{i=1}^{\infty} a_i^{(\nu)} = 0 \tag{20}$$

Then

$$\sum_{i=1}^{\infty} a_i^{(\nu)} = -1 \tag{21}$$

For the same assumption (16)

$$\lim_{i\to\infty} c_i^{(\nu)} = 1 \tag{22}$$

and

$$0 < c_i^{(\nu)} < c_{i+1}^{(\nu)} \tag{23}$$

2.2. Simplified Grünvald–Letnikov and Horner forms of the FOBD

Assuming that for some L

$$a_i^{(\nu)} \cong 0 \quad \text{for} \quad i \geq k-L+1, k-L+2, \cdots, k-1, k \tag{24}$$

whereas

$$c_i^{(\nu)} \cong 1 \quad \text{for} \quad i \geq k-L+1, k-L+2, \cdots, k-1, k \tag{25}$$

Then one can define a simplified Grünvald–Letnikov and Horner of the FOBD.

Definition 3. The simplified Grünvald–Letnikov FOBD of a discrete-time real, bounded function y_k is defined as a following sum

$$
{}_{0}^{GL}\Delta_{k,L}^{(\nu)} y_k = \sum_{i=0}^{k-L} a_i^{(\nu)} y_{k-i} \tag{26}
$$

Definition 4. The simplified Horner FOBD of a discrete-time real, bounded function y_k is defined as a following sum

$$
\begin{aligned}
&{}_{0}^{H}\Delta_{k,L}^{(\nu)} y_k \cong \\
&\cong c_0^{(\nu)}\left[y_k + c_1^{(\nu)}\left[y_{k-1} + c_2^{(\nu)}\left[y_{k-2} + \cdots + c_{k-L-1}^{(\nu)}\left[y_{L+1} + c_{k-L}^{(\nu)}\left[y_L + \cdots y_1 + y_0 \right]\right]\right]\cdots\right]\right]
\end{aligned} \tag{27}
$$

Formula (27) can be expressed as

$$
{}_{0}^{H}\Delta_{k,L}^{(\nu)} y_k = c_0^{(n)}\left[y_k + c_1^{(n)}\left[y_{k-1} + c_2^{(n)}\left[y_{k-2} + \cdots + c_{k-l}^{(n)}\left[y_l + \sum_{i=0}^{l-1} y_i \right]\right]\right]\cdots\right] \tag{28}
$$

In following numerical examples we show the influence of the “tail of a length L ” to an evaluation of a fractional order backward difference.

3. Numerical example

In this Section some FOBDs in the Grünvald–Letnikov and Horner forms of some discrete-time functions are numerically evaluated according to definition formulae (1) and (10) as well as to simplified forms.

First one considers the discrete-time unit step function

$$
y_k = \begin{cases} 0 & \text{for} \quad k < 0 \\ 1 & \text{for} \quad k \geq 0 \end{cases} \tag{29}
$$

In Figures 1a, b, c the FOBDs of order $\nu = 0.1$, 0.5 and 0.9 are presented. Figure 2 contains three plots. The first one (in black) is the ${}_{0}^{H}\Delta_{k}^{(0.5)} y_k$ $(= {}_{0}^{GL}\Delta_{k}^{(0.5)} y_k)$, the second one (in blue) is ${}_{0}^{GL}\Delta_{k,500}^{(0.5)} y_k$ and the last one (in red) is ${}_{0}^{H}\Delta_{k,500}^{(0.5)} y_k$.

To expose the values of errors in Figure 3 two error plots ${}_{0}^{H}\Delta_{k}^{(0.5)} y_k - {}_{0}^{GL}\Delta_{k,500}^{(0.5)} y_k$ (blue plot) and ${}_{0}^{H}\Delta_{k}^{(0.5)} y_k - {}_{0}^{H}\Delta_{k,500}^{(0.5)} y_k$ (red plot) are presented.

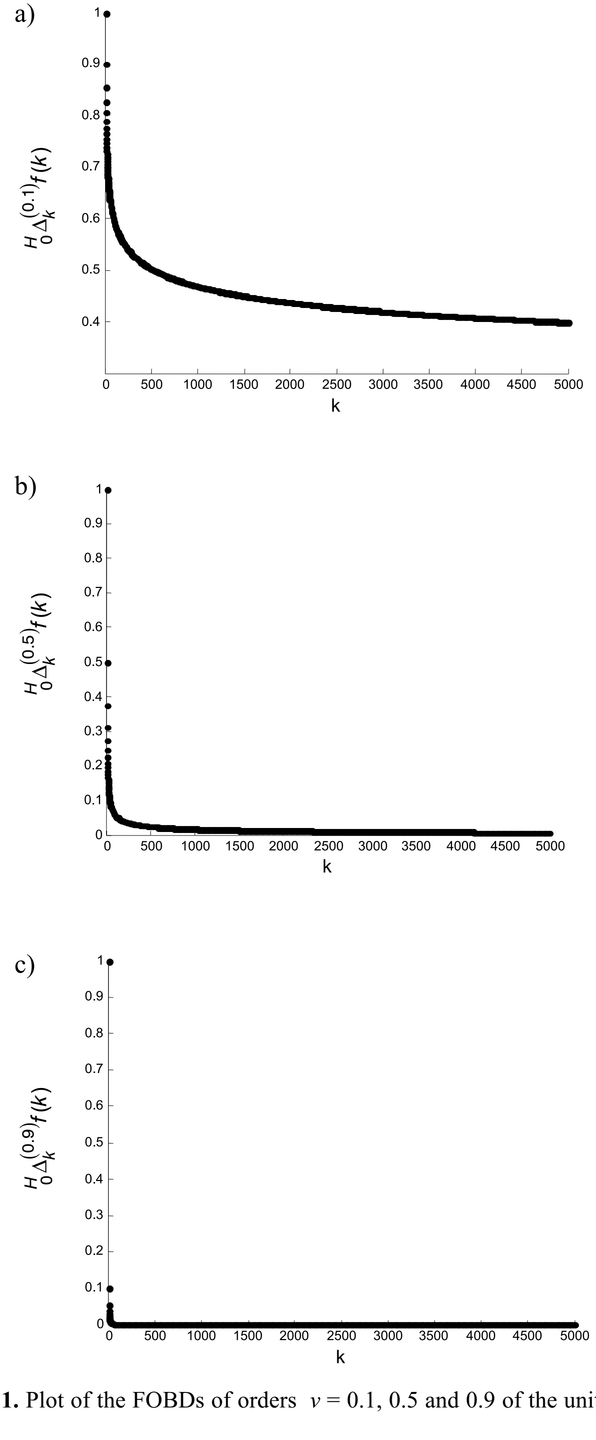

Fig. 1. Plot of the FOBDs of orders $v = 0.1$, 0.5 and 0.9 of the unit step function

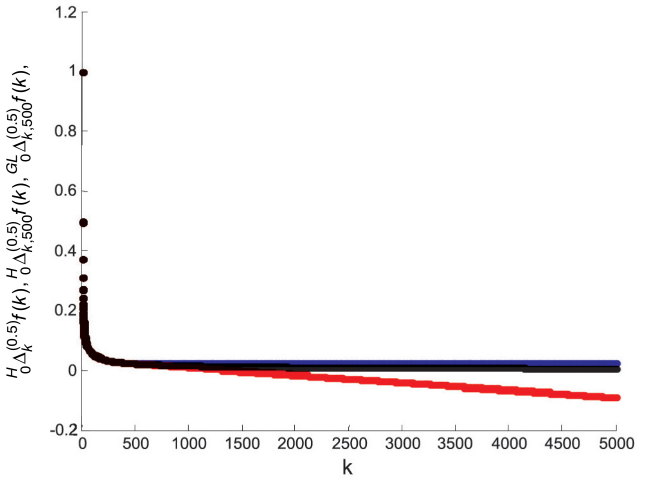

Fig. 2. Plot of the ${}^{H}_{0}\Delta^{(0.5)}_{k} y_k$ (= ${}^{GL}_{0}\Delta^{(0.5)}_{k} y_k$ (in black), the ${}^{GL}_{0}\Delta^{(0.5)}_{k,500} y_k$ (in blue) and the ${}^{H}_{0}\Delta^{(0.5)}_{k,500} y_k$ (in red)

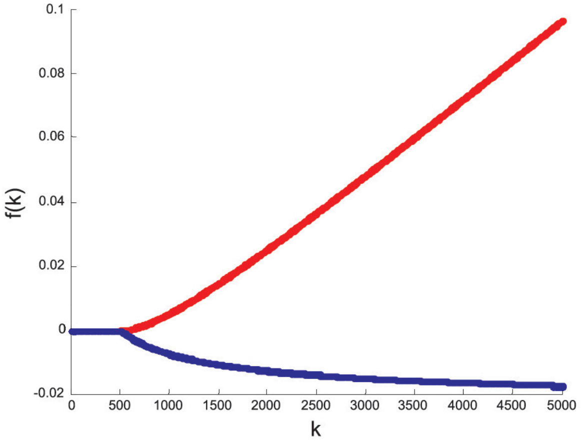

Fig. 3. Plot of the ${}^{H}_{0}\Delta^{(0.5)}_{k} y_k - {}^{GL}_{0}\Delta^{(0.5)}_{k,500} y_k$ (in blue) and the ${}^{H}_{0}\Delta^{(0.5)}_{k} y_k - {}^{H}_{0}\Delta^{(0.5)}_{k,500} y_k$ (in red)

Next one considers the staircase function defined below

$$f(k) = \left[\frac{k}{500} \right] \quad \text{for} \quad k = 0, 1, 2, ... \tag{30}$$

The plot of the considered function is given in Figure 4 whereas its FOBD ${}^{H}_{0}\Delta^{(0.5)}_{k} y_k$ in Figure 5.

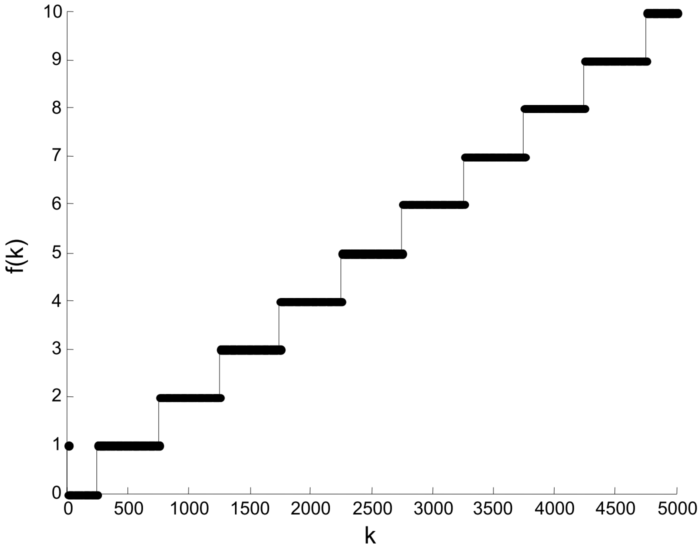

Fig. 4. Plot of the staircase function (30)

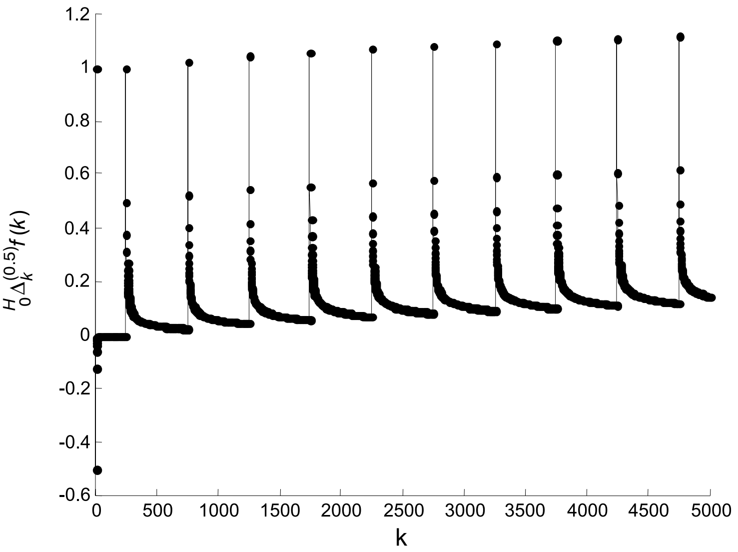

Fig. 5. Plot of the FOBD ${}^{H}_{0}\Delta^{(0.5)}_{k} y_k$ of the staircase function (30)

In Figure 6 the errors between simplified FOBD simplified forms the ${}^{H}_{0}\Delta^{(0.5)}_{k} y_k - {}^{GL}_{0}\Delta^{(0.5)}_{k,500} y_k$ (in blue) and the ${}^{H}_{0}\Delta^{(0.5)}_{k} y_k - {}^{H}_{0}\Delta^{(0.5)}_{k,500} y_k$ (in red) are presented.

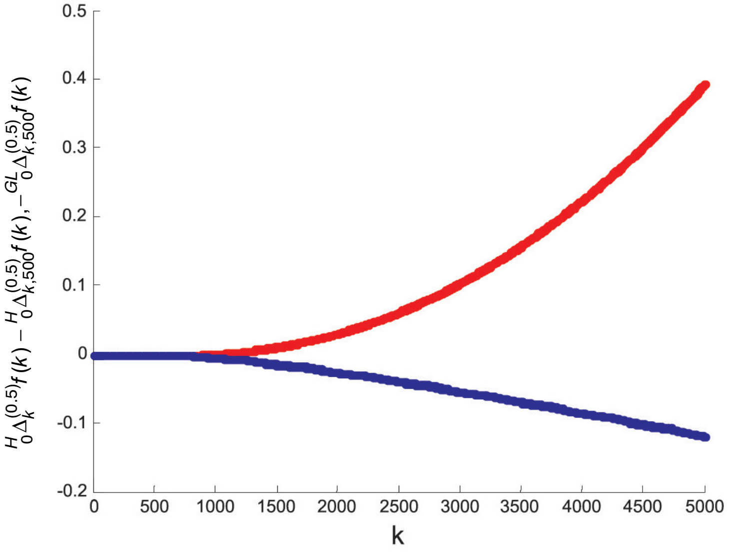

Fig. 6. Plot of the ${}_{0}^{H}\Delta_{k}^{(0.5)}y_k - {}_{0}^{GL}\Delta_{k,500}^{(0.5)}y_k$ (in blue) and the ${}_{0}^{H}\Delta_{k}^{(0.5)}y_k - {}_{0}^{H}\Delta_{k,500}^{(0.5)}y$ (in red)

Finally a periodic function plotted in Figure 7 is considered

$$f(k) = \sum_{i=1}^{4} \cos(i0.01k) \tag{31}$$

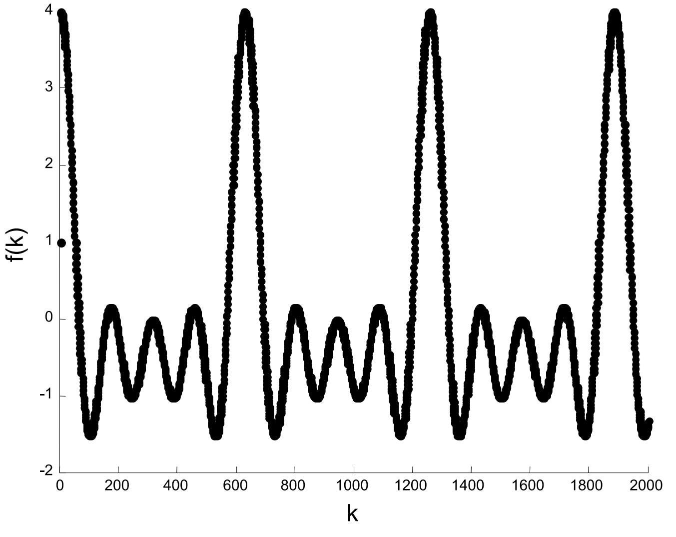

Fig. 7. Plot of the periodic function (31)

The FOBD ${}_{0}^{H}\Delta_{k}^{(0.5)}y_{k}$ of the periodic function (31) is given in Figure 8.

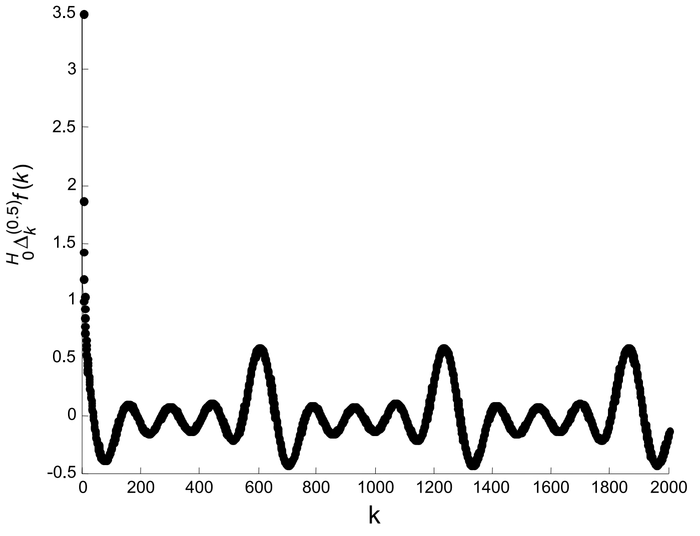

Fig. 8. Plot of the FOBD ${}_{0}^{H}\Delta_{k}^{(0.5)}y_{k}$ of the periodic function (31)

In Figure 9 the errors between simplified FOBD simplified forms the ${}_{0}^{H}\Delta_{k}^{(0.5)}y_{k} - {}_{0}^{GL}\Delta_{k,500}^{(0.5)}y_{k}$ (in blue) and the ${}_{0}^{H}\Delta_{k}^{(0.5)}y_{k} - {}_{0}^{H}\Delta_{k,500}^{(0.5)}y_{k}$ (in red) are presented.

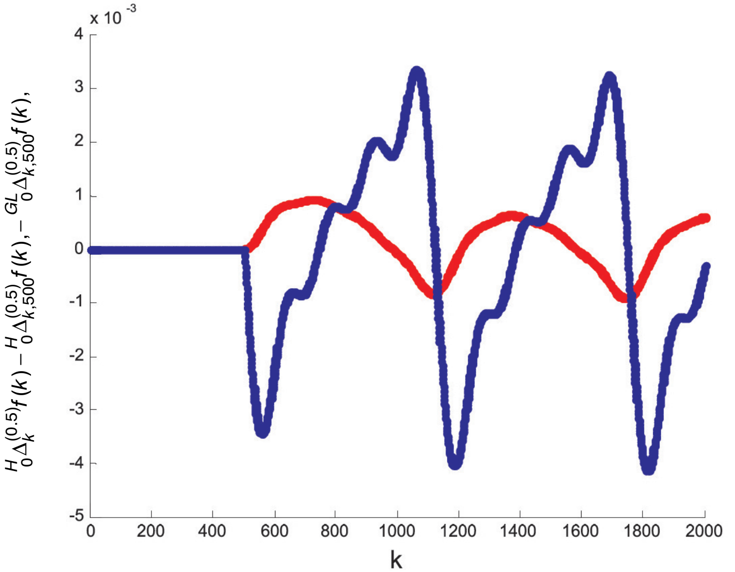

Fig. 9. Plot of the ${}_{0}^{H}\Delta_{k}^{(0.5)}y_{k} - {}_{0}^{GL}\Delta_{k,500}^{(0.5)}y_{k}$ (in blue) and the ${}_{0}^{H}\Delta_{k}^{(0.5)}y_{k} - {}_{0}^{H}\Delta_{k,500}^{(0.5)}y_{k}$ (in red)

4. Conclusions

In practical applications, for instance in the FO PID discrete-time controllers [2], one may apply simplified versions of the Grünvald–Letnikov or Horner forms of the FOBD. The numerical simulations show that the errors defined as

$$E_{GL}\left(\nu, L, y_k\right) = {}_{0}^{GL}\Delta_{k}^{(\nu)} y_k - {}_{0}^{GL}\Delta_{k,L}^{(\nu)} y_k \tag{32}$$

$$E_{H}\left(\nu, L, y_k\right) = {}_{0}^{H}\Delta_{k}^{(\nu)} y_k - {}_{0}^{H}\Delta_{k,L}^{(\nu)} y_k \tag{33}$$

depend on the FO, the length of "the calculation tail" as well as the type of function. Hence a calculation method and the FOBD evaluation device memory size and speed should be fitted to the function y_k to minimize errors.

References

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