Determination of thermal-stressed state of inhomogeneous orthotropic cylindrical shell under thermal heating

Roman Musii ORCID ID: 0000-0002-7169-2206

Lviv Polytechnic National University
Institute of Applied Mathematics and Fundamental Sciences
Department of Mathematics, Lviv
email: roman.s.musii@lpnu.ua

Nataliia Melnyk ORCID ID: 0000-0003-2337-2395

Lviv Polytechnic National University
Institute of Computer Science and Information Technologies
Department of Software, Lviv
email: nataliia.b.melnyk@lpnu.ua

Uliana Zhydyk ORCID ID: 0000-0002-1104-1946

Lviv Polytechnic National University
Institute of Applied Mathematics and Fundamental Sciences
Department of Mathematics, Lviv
email: uliana.v.zhydyk@lpnu.ua

Mykhaylo Melnyk ORCID ID: 0000-0002-8593-8799

Lviv Polytechnic National University
Institute of Computer Science and Information Technologies
Department of Computer-Aided Design Systems, Lviv
email: mykhaylo.r.melnyk@lpnu.ua

Oksana Oryshchyn ORCID ID: 0000-0002-8965-1891

Lviv Polytechnic National University
Institute of Applied Mathematics and Fundamental Sciences
Department of Mathematics, Lviv
email: oksana.h.oryshchyn@lpnu.ua

Krzysztof Pytel ORCID ID: 0000-0002-1924-8351

AGH University of Science and Technology
Faculty of Mechanical Engineering and Robotics
Department of Power Systems and Environmental Protection Facilities, Krakow
email: kpytel@agh.edu.pl

ABSTRACT

Systems of initial equations of the nonstationary problem of heat conductivity and the quasistatic problem of thermoelasticity for an inhomogeneous orthotropic cylindrical shell under its thermal heating by the external environment are recorded. Using double finite integral Fourier transforms in spatial coordinates and the Laplace transform over time, we obtained general solutions to the formulated thermoelasticity problems for a given finite hinged supported at the shell edges.

Keywords: thermal conductivity, thermoelasticity, cylindrical orthotropic shell, thermal heating

1. Introduction

Inhomogeneous cylindrical shells (in particular, those of layered structures) are widely used in many fields of modern technology (including aerospace construction) to increase the strength and rigidity of structures as well as protect them from low- or high-temperature heat. Therefore, the calculation of the thermal stress state of such structures is an important engineering task.

The elements of constructions of a layered structure have been considered in many works; particularly, in Reddy (2004) and Hetnarski (2014). Refined models that take the characteristics of composite materials into account (high anisotropy in the transverse direction in particular) were developed in Punera et al. (2018). Using the equation of interconnected thermoelasticity, the influence of the coefficient of cohesion on the dynamic behavior of composite shells was analyzed (Brischetto, Carrera 2010). The purpose of this section is to write down the systems of the initial equations of the nonstationary heat conductivity problem and the quasi-static thermoelasticity problem for inhomogeneous orthotropic cylindrical shells and to develop a method for constructing their general solutions under thermal heating by the external environment.

2. Formulation of problem and system of initial equations

Consider an inhomogeneous orthotropic circular cylindrical shell with a thickness of $2h$ and a finite length of l. The points of space of the shell are assigned to cylindrical coordinate system x, θ, z (which denotes the axial, circular, and radial coordinates, respectively). The origin is in the middle surface of the radius R shell. In the future, these coordinates will correspond to indices 1, 2, 3.

The shell is affected by external forces, and it can be heated by internal heat sources and the external environment. To study the thermoelastic behavior of such a shell, we use a mathematical model with six degrees of freedom. This model is based on assumptions about the linear distribution of displacement vector $U_i(x, \theta, z, \tau)$ ($i = 1, 2, 3$) and temperature $t(x, \theta, z, \tau)$ over the thickness of the shell.

$$U_i(x,\theta,z,\tau) = u_i(x,\theta,\tau) + z\gamma_i(x,\theta,\tau) \quad (1)$$

$$t(x,\theta,z,\tau) = T_1(x,\theta,\tau) + \frac{z}{h} T_2(x,\theta,\tau) \quad (2)$$

Here, u_i are the components of the displacement vector of the points of the middle surface, γ_i are the components of the vector of normal rotation angles, and T_n are integral characteristics of temperature:

$$T_n = \frac{2n-1}{2h^n} \int_{-h}^{h} t\, z^{n-1}\, dz, \; (n = 1, 2) \quad (3)$$

In the general case, this model of thermoelasticity of the considered shell consists of interconnected systems of equations of heat conductivity and thermoelasticity. If we neglect the effect of deformation on the change in the temperature field, these systems will be independent. Consider these systems in stages.

3. System of equations of nonstationary heat conductivity for inhomogeneous anisotropic shells

Let a thin shell of a constant thickness of $2h$ exchange heat with the environment according to Newton's law (or be heated by internal heat sources). The shell material is inhomogeneous in thickness and anisotropic (with one plane of heat symmetry). Its orthotropy axes coincide with the coordinate axes. The temperature field $t(\alpha, \beta, z, \tau)$ in such a shell is described by a three-dimensional heat-conduction equation, which takes the following form in a curvilinear orthogonal coordinate system α, β, z according to the accuracy of the theory of thin shells:

$$\Delta t + \frac{\partial}{\partial z}\left(\lambda_{33}(z)\frac{\partial t}{\partial z}\right) + 2k_0\lambda_{33}(z)\frac{\partial t}{\partial z} - c_e(z)\frac{\partial t}{\partial \tau} + w_t = 0 \quad (4)$$

where:

- w_t – density of heat sources,
- $c_e(z)$ – specific heat capacity,
- $\lambda_{ij}(z)$ – coefficients of heat conductivity,
- τ – time,
- A, B – corresponding Lamé coefficients,
- k_0 – average curvature of shell, a dimensionless quantity.

Here $\Delta = \frac{1}{AB}\left[\lambda_{11}(z)\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial}{\partial\alpha}\right)+\lambda_{22}(z)\frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial}{\partial\beta}\right)+2\lambda_{12}(z)\frac{\partial^2}{\partial\alpha\partial\beta}\right]$ – the Laplace operator in a curvilinear orthogonal coordinate system α, β, z.

To make the solution of Equation (1) unambiguous, we use initial condition $t|_{\tau=0} = t_0(\alpha,\beta)$. Here, $t_0(\alpha, \beta)$ is a function that describes the initial temperature distribution in the shell. As boundary conditions, we use one of the $t = t^{\pm}$, $\lambda_{33}(\partial t/\partial z) = \mp q^{\pm}$, $\lambda_{33}(\partial t/\partial z) \pm \alpha_z^{\pm}\left(t - t_z^{\pm}\right) = 0$ conditions on the $z = \pm h$ surfaces of the shell. Here, $\alpha_z^{\pm}$ are the coefficients of the heat transfer from the $z = \pm h$ surfaces, and $t^{\pm}$, $q^{\pm}$, $t_z^{\pm}$ are the temperature, heat flux, and temperature of the environment, respectively, that are set on these surfaces. If the shell is not closed, then similar conditions must be set at its edges. To formulate a two-dimensional problem on the integral characteristics of temperature, it is necessary to set the temperature distribution law over the shell thickness with the further use of the method of averaging for the original Equation (4) according to Formula (3).

For the cubic temperature distribution over the shell thickness for its integral characteristics T_1 and T_2 we obtain the following system of equations:

$$\Delta_{(1)}T_1 + \Delta_{(6)}T_2 + \frac{5k_0}{2h}\left(\Lambda_{33}^{(1)} - \Lambda_{33}^{(3)}\right)T_2 - C^{(1)}\dot{T}_1 - \frac{5}{12}\left(3C^{(2)} - C^{(4)}\right)\dot{T}_2 + W_t^{(1)} = f_1,$$

$$\Delta_{(2)}T_1 + \Delta_{(7)}T_2 + \frac{5k_0}{2h}\left(\Lambda_{33}^{(2)} - \Lambda_{33}^{(4)}\right)T_2 - \frac{5}{4h^2}\left(\Lambda_{33}^{(1)} - \Lambda_{33}^{(3)}\right)T_2 - \tag{5}$$

$$-C^{(2)}\dot{T}_1 - \frac{5}{12}\left(3C^{(3)} - C^{(5)}\right)\dot{T}_2 + W_t^{(2)} = f_2$$

Accordingly, the system of equations takes the following form for a linear temperature distribution for its integral characteristics T_1 and T_2:

$$\Delta_{(1)}T_1 + \Delta_{(2)}T_2 + \frac{2k_0}{h}\Lambda_{33}^{(1)}T_2 - C^{(1)}\dot{T}_1 - C^{(2)}\dot{T}_2 + W_t^{(1)} = F_1,$$

$$\Delta_{(2)}T_1 + \Delta_{(3)}T_2 + \frac{2k_0}{h}\Lambda_{33}^{(2)}T_2 - \frac{1}{h^2}\Lambda_{33}^{(1)}T_2 - C^{(2)}\dot{T}_1 - C^{(3)}\dot{T}_2 + W_t^{(2)} = F_2 \tag{6}$$

Here

$$\left\{\Lambda_{ij}^{(n)},\Lambda_{33}^{(n)},C^{(n)}\right\}=\int\limits_{-h}^{h}\left\{\lambda_{ij}(z),\lambda_{33}(z),c_e(z)\right\}\left(\frac{z}{h}\right)^{n-1}dz,\ (n=1,2,...,5),$$

$$\Delta_{(k)}=\frac{1}{AB}\left[\Lambda_{11}^{(k)}\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial}{\partial\alpha}\right)+\Lambda_{22}^{(k)}\frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial}{\partial\beta}\right)+2\Lambda_{12}^{(k)}\frac{\partial^2}{\partial\alpha\partial\beta}\right],(k=1,2,...,7), \quad (7)$$

$$\Lambda_{ij}^{(6)}=\frac{5}{12}\left(3\Lambda_{ij}^{(2)}-\Lambda_{ij}^{(4)}\right);\ \Lambda_{ij}^{(7)}=\frac{5}{12}\left(3\Lambda_{ij}^{(3)}-\Lambda_{ij}^{(5)}\right);\ W_i^t=\int\limits_{-h}^{h}w_t\left(\frac{z}{h}\right)^{i-1}dz;\ \dot{T}_i=\frac{\partial T_i}{\partial\tau}$$

where: $F(\alpha,\beta,\tau)_1, F_2(\alpha,\beta,\tau), f_1(\alpha,\beta,\tau), f_2(\alpha,\beta,\tau)$ are functions that depend on boundary conditions on $z=\pm h$ surfaces of shell.

Note that Equations (5) that correspond to the cubic law of temperature distribution are obtained under the condition that the boundary conditions on the $z=\pm h$ surfaces are satisfied and have the same order as in Equation (6). To determine the integral $\Lambda_{ij}^{(n)}, \Lambda_{33}^{(n)}, C^{(n)}$ characteristics of the thermophysical properties of an inhomogeneous material using Formula (7), it is necessary to specify the distribution law of these properties over the thickness.

In a cylindrical x, θ, z coordinate system, the heat conduction Equations (6) take the following form with respect to the integral characteristics of the temperature in the absence of heat sources:

$$\begin{aligned}&\left(\Delta_{(1)}-\varepsilon_1^t\right)T_1+\left(\Delta_{(2)}+\frac{\Lambda_{33}^{(1)}}{hR}-\varepsilon_2^t\right)T_2-C^{(1)}\dot{T}_1=-F_1^z,\\&\left(\Delta_{(2)}-\varepsilon_2^t\right)T_1+\left(\Delta_{(3)}-\frac{\Lambda_{33}^{(1)}}{h^2}-\varepsilon_1^t\right)T_2-C^{(3)}\dot{T}_2=-F_2^z\end{aligned} \quad (8)$$

Here, $\Delta_{(k)}=\Lambda_{11}^{(k)}\frac{\partial^2}{\partial x^2}+\frac{\Lambda_{22}^{(k)}}{R^2}\frac{\partial^2}{\partial\theta^2}$, $(k=1,2,3)$; $F_1^z=\varepsilon_1^t t_1^z+\varepsilon_2^t t_2^z+W_1^t$;

$F_2^z=\varepsilon_2^t t_1^z+\varepsilon_1^t t_2^z+W_2^t$.

To unambiguously solve the system of Equations (8), let us set the following boundary conditions for the integral T_1 and T_2 characteristics at the $x=0$ and $x=l$ edges of the shell:

$$T_1=T_2=0 \quad (9)$$

as well as the initial conditions at the $\tau = 0$ moment of time:

$$T_1(x,\theta,0) = T_1^0(x,\theta),\ T_2(x,\theta,0) = T_2^0(x,\theta) \tag{10}$$

4. Technique for solving problem of heat conductivity

After applying the double finite integral Fourier transform in the x, θ coordinates according to the specified boundary conditions in Equation (9), the system of Equations (8) will take the following form:

$$\frac{dT_{1mn}}{d\tau_1} + g_1 T_{1mn} + g_2 T_{2mn} = F_{1mn}^z,\ \frac{dT_{2mn}}{d\tau_1} + g_3 T_{1mn} + g_4 T_{2mn} = F_{2mn}^z \tag{11}$$

Here, $g_1 = L_{11}^{(1)}\mu_n^2 + L_{22}^{(1)}\delta^2 m^2 + Bi_1;\ g_2 = L_{11}^{(2)}\mu_n^2 + L_{22}^{(2)}\delta^2 m^2 - \delta + Bi_2;\ \mu_n = \frac{\pi n h}{l};$

$\delta = \frac{h}{R};\ g_3 = \tilde{C}\left(L_{11}^{(2)}\mu_n^2 + L_{22}^{(2)}\delta^2 m^2 + Bi_2\right);\ g_4 = \tilde{C}\left(L_{11}^{(3)}\mu_n^2 + L_{22}^{(3)}\delta^2 m^2 + Bi_1 + 1\right);$

$\tau_1 = \frac{\Lambda_{33}^{(1)}}{h^2 C^{(1)}}\tau;\ \tilde{C} = \frac{C^{(1)}}{C^{(3)}};\ L_{ii}^{(j)} = \frac{\Lambda_{ii}^{(j)}}{\Lambda_{33}^{(1)}};\ Bi_i = \frac{\varepsilon_i^t h^2}{\Lambda_{33}^{(1)}};$

$F_{1mn}^z = Bi_1 t_{1mn}^z + Bi_2 t_{2mn}^z + W_{1mn}^t \frac{h^2}{\Lambda_{33}^{(1)}};\ F_{2mn}^z = \left(Bi_2 t_{1mn}^z + Bi_1 t_{2mn}^z + W_{2mn}^t \frac{h^2}{\Lambda_{33}^{(1)}}\right)\tilde{C}.$

The solution of the system of Equations (11) under the initial conditions in Equations (10) using the time τ integral Laplace transform will be written as follows:

$$T_1 = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{\substack{j=1\\k\neq j}}^{2} \frac{\sin\frac{\pi n x}{l}\cos m\theta}{p_j - p_k}\Big\{\left(p_j - g_4\right)Q_{1nm}Z_1^{(j)}(\tau) + g_2 Q_{2nm} Z_2^{(j)}(\tau) +$$
$$+ \left[\left(p_j - g_4\right)T_{1nm}^0 + g_2 T_{2nm}^0\right]\exp(-p_j\tau_1)\Big\},$$
$$\tag{12}$$
$$T_2 = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{\substack{j=1\\k\neq j}}^{2} \frac{\sin\frac{\pi n x}{l}\cos m\theta}{p_j - p_k}\Big\{\left(p_j - g_1\right)Q_{2nm}Z_2^{(j)}(\tau) + g_3 Q_{1nm} Z_1^{(j)}(\tau) +$$
$$+ \left[\left(p_j - g_1\right)T_{2nm}^0 + g_3 T_{1nm}^0\right]\exp(-p_j\tau_1)\Big\}$$

Here

$$\left\{Q_{inm}, T_{inm}^{0}\right\} = \frac{\varsigma}{\pi l}\int_{0}^{l}\int_{-\pi}^{\pi}\left\{Q_i, T_i^{0}\right\}(x,\theta)\sin\frac{\pi n}{l}x\cos m\theta\, dx\, d\theta,\ \varsigma = \begin{cases}1, & m=0\\ 2, & m\neq 0\end{cases} \tag{13}$$

$$p_j = \frac{g_1+g_4}{2} + (-1)^j\sqrt{\frac{(g_1-g_4)^2}{4} + g_2 g_3}\ Z_i^{(j)} = \int_{0}^{\tau_1}\tilde{F}_i(u)\exp\left(-p_j(\tau_1-u)\right)du, \tag{14}$$

$(i,\ j = 1, 2)$

Taking Equations (13) and (14) into account, we obtain a general solution of the heat conductivity problem by substituting the expressions in Formula (12) into the linear law of temperature distribution over the entire thickness of the considered shell. Note that, using the same technique for the system of Equations (5), we wrote down the general solution of the heat conductivity problem for a given shell and in the case of a cubic law of temperature distribution over its thickness.

5. System of equations of thermoelasticity

The kinematic relationships for the e_{ij} deformation components of an arbitrary point of the shell are as follows:

$$e_{11} = \varepsilon_{11} + z\text{æ}_{11},\ e_{22} = \frac{\varepsilon_{22} + z\text{æ}_{22}}{1+\frac{z}{R}},\ e_{33} = \varepsilon_{33},$$
$$e_{12} = \frac{\varepsilon_{12} + z\text{æ}_{12} + z^2\omega_{12}}{1+\frac{z}{R}},\ e_{13} = \varepsilon_{13} + z\text{æ}_{13},\ e_{23} = \frac{\varepsilon_{23} + z\text{æ}_{23}}{1+\frac{z}{R}} \tag{15}$$

Here, the components of the deformation of the e_{ij}, æ_{ij} middle surface in terms of generalized displacements u_i, γ_i are described by the following formulas:

$$\varepsilon_{11} = \partial_1 u_1,\ \varepsilon_{22} = \frac{1}{R}(u_3 + \partial_2 u_2),\ \varepsilon_{33} = \gamma_3,\ \varepsilon_{12} = \frac{1}{R}\partial_2 u_1 + \partial_1 u_2,$$
$$\varepsilon_{23} = \gamma_2 + \frac{1}{R}(\partial_2 u_3 - u_2),\ \varepsilon_{13} = \gamma_1 + \partial_1 u_3,\ \omega_{12} = \frac{1}{R}\partial_1\gamma_2,\ \text{æ}_{11} = \partial_1\gamma_1, \tag{16}$$
$$\text{æ}_{22} = \frac{1}{R}(\gamma_3 + \partial_2\gamma_2),\ \text{æ}_{13} = \partial_1\gamma_3,\ \text{æ}_{12} = \partial_1\gamma_2 + \frac{1}{R}(\partial_2\gamma_1 + \partial_1 u_2),\ \text{æ}_{23} = \frac{1}{R}\partial_2\gamma_3$$

The physical equations for the stresses and strains take the following form:

$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\end{pmatrix}=\begin{pmatrix}c_{11} & c_{12} & c_{13} & \\ c_{12} & c_{22} & c_{23} & \\ c_{13} & c_{23} & c_{33} & \\ & & & c_{66}\end{pmatrix}\begin{pmatrix}e_{11}\\ e_{22}\\ e_{33}\\ e_{12}\end{pmatrix}-\begin{pmatrix}\beta_{11}^{t}\\ \beta_{22}^{t}\\ \beta_{33}^{t}\\ \end{pmatrix}t,\quad \begin{pmatrix}\sigma_{13}\\ \sigma_{23}\end{pmatrix}=\begin{pmatrix}c_{44} & \\ & c_{55}\end{pmatrix}\begin{pmatrix}e_{13}\\ e_{23}\end{pmatrix} \tag{17}$$

where:

$c_{ij}(z)$ – coefficients of elasticity,

$\beta_{ii}^{t}(z)=c_{i1}\alpha_{11}^{t}+c_{i2}\alpha_{22}^{t}+c_{i3}\alpha_{33}^{t}$ – coefficients of thermoelasticity,

$\alpha_{ij}^{t}(z)$ – coefficients of thermal linear expansion.

The physical equations for internal efforts N_{ij} and moments M_{ij} are obtained from the following relationships:

$$\begin{aligned}
&\{N_{11},N_{12},N_{13}\}=\int_{-h}^{h}\{\sigma_{11},\sigma_{12},\sigma_{13}\}\left(1+\frac{z}{R}\right)dz,\\
&\{N_{22},N_{21},N_{23}\}=\int_{-h}^{h}\{\sigma_{22},\sigma_{12},\sigma_{23}\}\,dz,\\
&\{M_{11},M_{12},M_{13}\}=\int_{-h}^{h}\{\sigma_{11},\sigma_{12},\sigma_{13}\}\left(1+\frac{z}{R}\right)z\,dz,\\
&\{M_{22},M_{21},M_{23}\}=\int_{-h}^{h}\{\sigma_{22},\sigma_{12},\sigma_{23}\}\,z\,dz,\ N_{33}=\int_{-h}^{h}\sigma_{33}\left(1+\frac{z}{R}\right)dz
\end{aligned} \tag{18}$$

The equilibrium equations will be as follows:

$$\begin{aligned}
&\partial_1 N_{11}+\frac{1}{R}\partial_2 N_{21}=-q_1,\ \partial_1 N_{12}+\frac{1}{R}(\partial_2 N_{22}+N_{23})-q_2,\\
&\partial_1 N_{13}+\frac{1}{R}(\partial_2 N_{23}-N_{22})=-q_3,\\
&\partial_1 M_{11}+\frac{1}{R}\partial_2 M_{21}-N_{13}=-m_1,\ \partial_1 M_{12}+\frac{1}{R}\partial_2 M_{22}-N_{23}=-m_2,\\
&\partial_1 M_{13}+\frac{1}{R}(\partial_2 M_{23}-M_{22})-N_{33}=-m_3
\end{aligned} \tag{19}$$

Here, q_i, m_i are the components of the $\partial_1 = \partial/\partial x$ and $\partial_2 = \partial/\partial\theta$ surface loads. Using the above relationships, we write down a system of equilibrium Equations (19) in generalized displacements:

$$\sum_{k}^{6} L_{rk} y_k = b_r \quad (r, k = 1, 2, ..., 6) \tag{20}$$

Here, $y_i = u_i$; $y_{3+i} = \gamma_i$ (i = 1, 2, 3)are generalized movements. Differential operators L_{rk} ($L_{rk} = L_{kr}$)and members b_r in system of Equations (20) take the following form:

$$L_{11} = A_{11}\partial_{11}^2 + \frac{A_{66}}{R^2}\partial_{22}^2,\ L_{12} = \frac{A_{12} + A_{66}}{R}\partial_{12}^2,\ L_{13} = \frac{A_{12}}{R}\partial_1,\ L_{14} = B_{11}\partial_{11}^2 + \frac{B_{66}}{R^2}\partial_{22}^2,$$

$$L_{15} = \frac{B_{12} + B_{66}}{R}\partial_{12}^2,\ L_{16} = \left(A_{13} + \frac{B_{12}}{R}\right)\partial_1,\ L_{22} = A_{66}\,\partial_{11}^2 + \frac{1}{R^2}\left(A_{22}\partial_{22}^2 - k'A_{55}\right),$$

$$L_{23} = \frac{A_{22} + k'A_{55}}{R^2}\partial_2,\ L_{24} = \frac{B_{12} + B_{66}}{R}\partial_{12}^2,\ L_{25} = B_{66}\partial_{11}^2 + \frac{B_{22}}{R^2}\partial_{22}^2 + \frac{k'A_{55}}{R},$$

$$L_{26} = \left(\frac{A_{23}}{R} + \frac{B_{22} + k'B_{55}}{R^2}\right)\partial_2,\ L_{33} = -k'A_{44}\partial_{11}^2 - \frac{k'A_{55}}{R^2}\left(k'A_{55}\partial_{22}^2 + A_{22}\right),$$

$$L_{34} = \left(\frac{B_{12}}{R} - k'A_{44}\right)\partial_1,\ L_{35} = \frac{1}{R}\left(\frac{B_{22}}{R} - k'A_{55}\right)\partial_2,$$

$$L_{36} = -k'B_{44}\partial_{11}^2 + \frac{1}{R^2}\left(B_{22} - k'B_{55}\partial_{22}^2\right) + \frac{A_{23}}{R},$$

$$L_{44} = D_{11}\partial_{11}^2 + \frac{D_{66}}{R^2}\partial_{22}^2 - k'A_{44},\ L_{45} = \frac{D_{12} + D_{66}}{R}\partial_{12}^2,\ L_{46} = \left(\frac{D_{12}}{R} + B_{13} - k'B_{44}\right)\partial_1,$$

$$L_{55} = D_{66}\partial_{11}^2 + \frac{D_{22}}{R^2}\partial_{22}^2 - k'A_{55},\ L_{56} = \frac{1}{R}\left(B_{23} - k'B_{55} + \frac{D_{22}}{R}\right)\partial_2,$$

$$L_{66} = A_{33} + \frac{2B_{23}}{R} + \frac{1}{R^2}\left(D_{22} - k'D_{55}\partial_{22}^2\right) - k'D_{44}\partial_{11}^2,\ b_1 = A_{11}^t\partial_1 T_1 + \frac{B_{11}^t}{h}\partial_1 T_2 - q_1,$$

$$b_2 = \frac{A_{22}^t}{R}\partial_2 T_1 + \frac{B_{22}^t}{Rh}\partial_2 T_2 - q_2,\ b_3 = \frac{A_{22}^t}{R}T_1 + \frac{B_{22}^t}{Rh}T_2 + q_3,$$

$$b_4 = B_{11}^t\partial_1 T_1 + \frac{D_{11}^t}{h}\partial_1 T_2 - m_1,\ b_5 = \frac{B_{22}}{R}\partial_2 T_1 + \frac{D_{22}^t}{Rh}\partial_2 T_2 - m_2.$$

Here, $\left\{A_{ii}, B_{ij}, D_{ij}\right\} = \int\limits_{-h}^{h} c_{ij} \left\{1, z, z^2\right\} dz, \left\{A_{ii}^t, B_{ii}^t, D_{ii}^t\right\} = \int\limits_{-h}^{h} \beta_{ii}^t \left\{1, z, z^2\right\} dz$ (k'– shear coefficient).

For the solution of system of Equations (20) to be unambiguous, it is necessary to set the appropriate boundary conditions. In the case of a cylindrical shell of a finite length, one value from each of the following pairs are set at its and ends:

$$\{N_{11}, u_1\}; \{N_{12}, u_2\}; \{N_{13}, u_3\}; \{M_{11}, \gamma_1\}; \{M_{12}, \gamma_2\}; \{M_{13}, \gamma_3\}.$$

The system of Equations (20) together with the boundary conditions form the boundary value problem of quasi-static thermoelasticity for inhomogeneous anisotropic cylindrical shells in displacements. According to the known displacements from the relationships in Formula (16), we determine the deformations of the mean surface and the forces and moments according to the equations is:

$$\begin{pmatrix} N_{11} \\ N_{22} \\ N_{33} \\ M_{11} \\ M_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & B_{11} & B_{12} \\ A_{12} & A_{22} & A_{23} & B_{12} & B_{22} \\ A_{13} & A_{23} & A_{33} & B_{13} & B_{23} \\ B_{11} & B_{12} & B_{13} & D_{11} & D_{12} \\ B_{12} & B_{22} & B_{23} & D_{12} & D_{22} \end{pmatrix} \begin{pmatrix} \partial_1 u_1 \\ (\partial_2 u_2 + u_3)/R \\ \gamma_3 \\ \partial_1 \gamma_1 \\ (\partial_2 \gamma_2 + \gamma_3)/R \end{pmatrix} - \begin{pmatrix} A_{11}^t \\ A_{22}^t \\ A_{33}^t \\ B_{11}^t \\ B_{22}^t \end{pmatrix} T_1 - \begin{pmatrix} B_{11}^t \\ B_{22}^t \\ B_{33}^t \\ D_{11}^t \\ D_{22}^t \end{pmatrix} T_2/h, \tag{21}$$

$$\begin{pmatrix} N_{12} \\ M_{12} \end{pmatrix} = \begin{pmatrix} A_{66} & B_{66} \\ B_{66} & D_{66} \end{pmatrix} \begin{pmatrix} \partial_1 u_2 + \partial_2 u_1/R \\ \partial_1 \gamma_2 + \partial_2 \gamma_1/R \end{pmatrix}, \begin{pmatrix} N_{13} \\ M_{13} \end{pmatrix} = k' \begin{pmatrix} A_{44} & B_{44} \\ B_{44} & D_{44} \end{pmatrix} \begin{pmatrix} \gamma_1 + \partial_1 u_3 \\ \partial_1 \gamma_3 \end{pmatrix},$$

$$\begin{pmatrix} N_{23} \\ M_{23} \end{pmatrix} = k' \begin{pmatrix} A_{55} & B_{55} \\ B_{55} & D_{55} \end{pmatrix} \begin{pmatrix} \gamma_2 + (\partial_2 u_3 - u_2)/R \\ \partial_2 \gamma_3 / R \end{pmatrix}$$

According to the obtained forces and moments, Formulas (15) and (17), we write down the temperature stresses and strains in the cylindrical shell.

6. Methods for solving thermoelasticity problem

Consider a cylindrical shell that is antisymmetric relative to the middle surface and composed of an even number of orthotropic layers with the same thickness and properties – the material axes of which are oriented at angles of 0° or 90° to the axis of the shell. Let the $x = 0$ and $x = l$ edges of the shells be hinged, and a temperature of 0°C is set on them.

Then, we would have the following boundary conditions:

$$u_3 = u_2 = 0;\ \gamma_3 = \gamma_2 = 0;\ N_{11} = M_{11} = 0 \tag{22}$$

$$T_1 = T_2 = 0 \tag{23}$$

At the initial moment, the temperature characteristics are set by the coordinate functions:

$$T_1(x,\theta,0) = T_1^0(x,\theta),\ T_2(x,\theta,0) = T_2^0(x,\theta) \tag{24}$$

The solution of the system of equilibrium Equations (20) that satisfies the boundary conditions in (22) for the known integral characteristics of temperatures T_1 and T_2 is obtained by means of finite integral Fourier transforms in the x, θ coordinates. As a result, we obtain a system of algebraic equations to determine the Fourier coefficients of the y_{kmn} required functions. We write this system of equations in matrix form:

$$\mathbf{AY} = \mathbf{S}T_{1mn} + \mathbf{G}T_{2mn} \tag{25}$$

where: corresponding matrices will be $\mathbf{A} = (a_{rk})_{6\times6}$, $\mathbf{Y} = (y_{kmn})_{6\times1}$, $\mathbf{S} = (s_k)_{6\times1}$, $\mathbf{G} = (g_k)_{6\times1}$.

Here, $y_{imn} = U_{imn}$ are the Fourier coefficients for displacements u_i, and $y_{3+i,mn} = \Gamma_{imn}$ are the Fourier coefficients for displacements γ_i ($i = 1, 2, 3$). The coefficients of the a_{rk}, s_k and g_k matrices are calculated using finite integral Fourier transforms in spatial coordinates to the expressions of the differential operators of the system of Equations (20).

After the transformations, we obtain the solution of system of Equations (25) in the following form:

$$y_{kmn} = \frac{1}{|A|}\sum_{r=1}^{6}(s_r T_{1mn} + g_r T_{2mn})B_{rk},\ (k = 1, 2, ..., 6)$$

Here, $|A|$ is the determinant of matrix **A**, and is an algebraic addition to the element a_{rk} of this matrix. The generalized displacements are written by the following expressions in terms of the corresponding Fourier coefficients:

$$\begin{aligned}
\{u_1, \gamma_1\} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\{U_{1mn}, \Gamma_{1mn}\}\cos\frac{\pi n}{l}x\cos m\theta, \\
\{u_2, \gamma_2\} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{U_{2mn}, \Gamma_{2mn}\}\sin\frac{\pi n}{l}x\sin m\theta, \\
\{u_3, \gamma_3\} &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\{U_{3mn}, \Gamma_{3mn}\}\sin\frac{\pi n}{l}x\cos m\theta
\end{aligned} \tag{26}$$

According to the known generalized displacements (26) and the temperature field, all of the other characteristics of the stress and strain state of the shell are determined by Formulas (15)–(17) and (21).

REFERENCES

Brischetto S., Carrera E., 2010, *Coupled thermo-mechanical analysis of one-layered and multilayered isotropic and composite shells*, Computer Modeling in Engineering and Sciences, 56(3), 249–301.

Hetnarski R.B. (Ed.), 2014, *Encyclopedia of Thermal Stresses*, Springer.

Punera D., Kant T., Desai Y., 2018, *Thermoelastic analysis of laminated and functionally graded sandwich cylindrical shells with two refined higher order models*, Journal of Thermal Stresses, 41(1), 54–79.

Reddy J.N., 2004, *Mechanics of Laminated Composite Plates and Shells. Theory and Analysis*, CRC Press, New York.