Opuscula Math. 34, no. 2 (2014), 443–456
http://dx.doi.org/10.7494/OpMath.2014.34.2.443 Opuscula Mathematica

Dedicated to the Memory of Professor Zdzisław Kamont

METHOD OF LINES FOR PARABOLIC STOCHASTIC FUNCTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Communicated by Bolesław Kacewicz

Abstract. We approximate parabolic stochastic functional differential equations substituting the derivatives in the space variable by finite differences. We prove the stability of the method of lines corresponding to a parabolic SPDE driven by Brownian motion.

Keywords: stochastic partial differential equations, stability of the method of lines, white noise, Volterra stochastic equations.

Mathematics Subject Classification: 35R60, 49M25.

1. INTRODUCTION

The method of lines is a semidiscrete numerical method. The idea is to discretize the spatial variable and reduce the given equation to a system of ordinary differential equations. The derivatives with respect to the space variable x are approximated by the finite difference quotients

$$\frac{\partial}{\partial x}u(t,x) \approx \frac{u(t,x+h)-u(t,x-h)}{2h},$$
$$\frac{\partial^2}{\partial x^2}u(t,x) \approx \frac{u(t,x+h)-2u(t,x)+u(t,x-h)}{h^2},$$

where $h>0$ is the discretization step. The most important property of the scheme is convergence. It means that the solution of the difference scheme approximates the solution of the corresponding differential equation and the approximation improves as the grid spacing h tends to zero.

The method of lines was widely used to discretize deterministic partial differential equations of various types. J. Bebernes, R. Ely and C. Boulder in [1] studied partial differential equations of parabolic type of the form

$$\frac{\partial u}{\partial t} - \Delta u = \delta e^u + ((\gamma - 1)/vol\Omega)\delta \int\limits_{\Omega} e^u dy.$$

J. Kauthen in [14] implemented MOL for the parabolic equation

$$\frac{\partial u}{\partial t} = g + \sum_{i=0}^{2} a_1 \frac{\partial^i u}{\partial x^i} + \int\limits_0^t b(x,t,s,u)ds, \quad 0 < x < 1,\ 0 < t \leq T.$$

It was also applied to the numerical solution of nonlinear Burgers'-type equations [12]

$$\frac{\partial u}{\partial t} + \alpha u^\delta \frac{\partial u}{\partial x} - \nu \frac{\partial u}{\partial x^2} = \beta(1-u^\delta)(u^\delta - \gamma), \quad a \leq x \leq b,\ t \geq 0.$$

The deterministic Fisher-Kolmogorov-Petrovsky-Piscounov (FKPP) of the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^2$$

appeared in [5] and [15]. It was first introduced in problems of genetics and then appeared in much broader contexts like reaction-diffusion problems or travelling-wave solutions.

The numerical method of lines was also applied to extended Boussinesq equations

$$\frac{\partial \eta}{\partial t} + c\frac{\partial u}{\partial x} + \frac{\partial \eta u}{\partial x} + (\alpha + 1/3)c^3 \frac{\partial^3 u}{\partial x^3} = 0,$$

$$\frac{\partial u}{\partial t} + g\frac{\partial \eta}{\partial x} + u\frac{\partial u}{\partial x} + \alpha c^2 \frac{\partial^3 u}{\partial t \partial x^2}$$

with some constants c, α, g (see [10]) or generalized Kuramoto-Sivashinsky equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^3 u}{\partial x^3} + \eta\frac{\partial^4 u}{\partial x^4} = 0, \quad a \leq x \leq b,\ t \geq 0$$

in [11]. A class of abstract differential equations is solved by MOL in [19]. The method of lines for delay differential equations of the form

$$\frac{\partial u}{\partial t} = f(t, u(t), u(t-\tau)), \quad t \geq 0,\ u(t) = \varphi(t),\ -r \leq t \leq 0$$

can be found in [16]. Applications of MOL to nonlinear nonlocal differential equations

$$\frac{\partial u}{\partial t} + Au(t) = f(t, u(t), u_t), \quad t \in (0,T],\ h(u_0) = \phi, \text{ on } [-\tau, 0]$$

are shown in [3]. The stability of MOL is shown in [21].

Discretization schemes for parabolic SPDE's driven by white noise have been considered by several authors. J.B. Walsh in [24] studies finite element methods for parabolic SPDE's

$$\begin{cases} \dfrac{\partial U}{\partial t} = \dfrac{\partial^2 U}{\partial x^2} + f(U)\dot{B} + g(U), & (x,t) \in [0,L] \times \mathbb{R}_+, \\ U(x,0) = u_o(x), & x \in [0,L], \\ U(0,t) = U(L,t) = 0, & t > 0. \end{cases}$$

I. Gyöngy [7] studies the strong convergence in the uniform norm over the space variable for a finite-difference scheme with a regular mesh on $[0,1]$ for the parabolic SPDE

$$\frac{\partial}{\partial t} u = \frac{\partial^2 u}{\partial x^2} + f(t,x,u) + \sigma(t,x,u)\frac{\partial^2}{\partial t \partial x} B$$

with Dirichlet boundary conditions. A. Rössler, M. Seaid and M. Zahri in [22] proposed the method of lines for stochastic initial boundary-value problems with additive noise

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} - v\frac{\partial^2 u}{\partial x^2} = \sigma(x)\zeta(t,x), \quad (t,x) \in (0,T] \times \mathcal{D},$$

where $\mathcal{D}$ is a bounded spatial domain and $\zeta(t,x)$ is a random noise assumed to be either time-dependent or space-dependent with amplitude σ. In practice, the random process $\zeta(t,x)$ is Gaussian with zero mean and statistically homogeneous with covariance $\langle \zeta(t,x)\zeta(s,x')\rangle = 2B(x-x')\delta(t-s)$ where $B(x)$ is a smooth function and δ is the Dirac function. An application of the stochastic parabolic PDE is a stochastic generalization of the FKPP equation with time dependent white noise of the form (see [17])

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \mu f(u) + g(u)\dot{B}$$

for $u = u(t,x)$, $(t,x) \in [0,\infty) \times \mathbb{R}$. The parameter $D > 0$ controls the diffusion and $\mu > 0$ can be interpreted as a growth rate. Another model of the stochastic FKPP of the form

$$\frac{\partial u}{\partial t} = \frac{\mu^2}{2}\frac{\partial u}{\partial x^2} + \frac{1}{\mu^2}u(1-u) + \hat{\epsilon} u \dot{B}$$

with one-dimensional time-dependent white noise $\dot{B}$, a small parameter $0 < \mu \ll 1$ and noise strenght $\hat{\epsilon} > 0$, can be found in Elworthy, Zhao and Gaines (see [4]). Stochastic FKPP with space-time white noise was considered by Doering, Mueller and Smereka in [2]. They deduced some properties of the solution for the equation of the form

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \gamma u(1-u) + \epsilon\sqrt{u(1-u)}\eta(t,x),$$

for $0 \le u \le 1$, where $\eta(t,x)$ is a Gaussian white noise in space and time.

The aim of this paper is to discuss the stability of the space discretization for the initial value problem for the stochastic partial differential equation which arise from the FKPP model

$$\frac{\partial}{\partial t}u(t,x)-\frac{\partial^2}{\partial x^2}u(t,x)=u(t,x)\left[1-\int_0^t u(s,x)ds\right]+\int_{-r}^{0}\int_{-\infty}^{\infty}K(t,s,y)u(t+s,y)dyds\dot{B}_t$$

driven by time dependent white noise. Note that the nonlocal coefficient of $\dot{B}_t$ is independent of x. We study the following generalization of the above modified FKPP equation

$$\frac{\partial}{\partial t}u(t,x)-\frac{\partial^2}{\partial x^2}u(t,x)=f(t,x,u_{(t,x)})+g(t,u_{(t,0)})\dot{B}.$$

It is crucial for the given function g to be independent of x, that is why we use the Hale operator at a fixed point $(t,0)$, which does not depend on x. The function g can be for example of the form

$$g(t,u_{(t,0)})=\tilde{g}\left(t,\int_0^1 u(0,y)dy\right),$$

$$g(t,u_{(t,0)})=\tilde{g}\left(t,\int_{-\infty}^{\infty} e^{-y^2}u(t,y)dy\right)$$

or

$$g(t,u_{(t,0)})=\int_{-r}^{0}\int_{-\infty}^{\infty}K(t,s,y)u(t+s,y)dyds.$$

In [23] Seidler quotes an example of a stochastic heat equation with a drift of the form

$$\frac{\partial u}{\partial t}(t,x)=\frac{\partial^2 u}{\partial x^2}(t,x)+p(u(t,x))+\dot{B}_t. \tag{1.1}$$

Thus our model is an extension of (1.1). Lord in [20] tests numerically a reaction-diffusion Allen-Cahn equation from mathematical physics with noise of the form

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial t}\left[\frac{\alpha\partial^2 u}{\partial x^2}+u-u^3\right]+\dot{B}_t,\quad u(0)=u_0.$$

We admit that time-dependent white noise causes much different qualitative effects than space-time white noise. The space-time white noise generates huge theoretical problems in unbounded domains and the results of this paper could not be extended to that case. Especially, the results from the work of Gyöngy [8] could not be extended to the unbounded case. Even Malliavin calculus and abstract white noise expansion do not seem to be sufficient to estimate the case with space-time white noise. We discretize derivatives with respect to the space variable and study the representation of the solution of the obtained infinite system of stochastic integral equations of Volterra type. The integral representation of the solution by Green's functions, derived in our paper, is useful for a priori estimates and error analysis. The effectiveness of that representation results from MOL analysis in [18] based on the maximum principle. There is no classical maximum principle for stochastic PDE's but we use Doob's martingale inequality to estimate the classical Itô integral. The technique used in this paper could not be extended to the case, where the function g depends on a spatial variable because this leads to Volterra-Itô integrals which are no longer martingales.

We have limited ourselves to one spatial dimension and the uniform mesh, although all the results remain true if we replace 1-D with $\mathbb{R}^n$ and the regular mesh with an irregular one. The results can be easily generalized to the multidimensional case. This work is the first step towards studying the convergence of the method of lines for more general models with space-time and functional dependence of the function $g = g(t, x, u_{(t,x)})$. This model leads to the stochastic convolution of the form $\int_0^t G(t, s, x, u)dB_s$ which causes other technical problems.

2. FORMULATION OF THE PROBLEM

Suppose $r \geq 0$, $T > 0$. Let $(B_t)_{t\in[0,T]}$ be the standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_t$ its natural filtration. Let $E = [0, T] \times \mathbb{R}$, $E_0 = [-r, 0] \times \mathbb{R}$, $E_T = [-r, T] \times \mathbb{R}$ and let C_{E_T} denote the space of all continuous and $\mathcal{F}_t$-adapted processes $w : E_T \to L^2(\Omega)$ with the finite norm,

$$\|w\|^2 = \mathbb{E}\left[\sup_{-r\leq t\leq T,\, x\in\mathbb{R}} |w^2(\tilde{t}, x)|\right] < \infty.$$

Then, for any $t \in [0, T]$, we define the Hale type operator (see [9])

$$u_{(t,x)}(\tau, \theta) = u(t + \tau, x + \theta) \quad \text{for} \quad (\tau, \theta) \in E_0.$$

Let $\dot{B}_t$ denote the formal derivative of a one-dimensional Brownian motion B_t. We consider the following initial value problem for stochastic functional partial differential equations

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) - \dfrac{\partial^2}{\partial x^2}u(t,x) = f(t,x,u_{(t,x)}) + g(t,u_{(t,0)})\dot{B}_t & \text{for } (t,x) \in E, \\ u(t,x) = \varphi(t,x) & \text{for } (t,x) \in E_0, \end{cases} \tag{2.1}$$

where

$$\varphi : E_0 \to \mathbb{R}, \quad f : E \times \mathcal{C}(E_0, \mathbb{R}) \to \mathbb{R}, \quad g : [0, T] \times C(E_0, \mathbb{R}) \to \mathbb{R} \tag{2.2}$$

and $C(E_0,\mathbb{R})$ denotes the space of all continuous functions $v : E_0 \to \mathbb{R}$. Mild solutions of (2.1) are continuous and satisfy the integral equation

$$\begin{aligned} u(t,x) = &\int\limits_{\mathbb{R}} G_t(x-y)u_0(y)dy + \\ &+ \int\limits_0^t \int\limits_{\mathbb{R}} G_{t-s}(x-y)f(s,y,u_{(s,y)})dsdy + \int\limits_0^t g(s,u_{(s,0)})dB_s, \end{aligned} \tag{2.3}$$

where the last integer is of the Itô type and $G_t(x-y)$ is the Green function or fundamental solution of the homogenous heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$. Suppose that the function f is continuous and satisfies the Lipschitz condition with respect to the third (functional) variable, so there exists a constant $L_1 > 0$ such that

$$|f(t,x,v) - f(t,x,\bar{v})| \le L_1 \sup_{(s,y)\in E_0} |v(s,y) - \bar{v}(s,y)| \tag{L1}$$

and

$$|f(t,x,0)| \le C_1 \tag{B1}$$

for all t, x, v with some constant C_1. Suppose that the function g is continuous and satisfies the Lipschitz condition with respect to the second (functional) variable, so there exists a constant $L_2 > 0$ such that

$$|g(t,v) - g(t,\bar{v})| \le L_2 \sup_{(s,y)\in E_0} |v(s,y) - \bar{v}(s,y)| \tag{L2}$$

and

$$|g(t,0)| \le C_2 \tag{B2}$$

for all t, v with some constant C_2.

The above conditions imply the existence and uniqueness for the Cauchy problem (2.1).

3. STABILITY OF THE METHOD OF LINES

Problem (2.1) is discretized in the spatial variable as follows. We introduce a uniform mesh on $\mathbb{R}$ with the discretization step $h > 0$. We will denote by J_h the piecewise linear interpolating operator (see [13]).

$$\begin{cases} \dfrac{d}{dt}u^{(i)}(t) - \dfrac{u^{(i+1)}(t) - 2u^{(i)}(t) + u^{(i-1)}(t)}{h^2} = \\ \qquad = f(t, h\cdot i, (J_h u)_{(t,h\cdot i)}) + g(t,(J_h u)_{(t,0)})\dot{B}_t & \text{for } t\in[0,T],\ i\in\mathbb{Z}, \\ u^{(i)}(t) = \varphi^{(i)}(t) & \text{for } i\in\mathbb{Z},\ t\in[-r,0]. \end{cases} \tag{3.1}$$

For processes $W = (w^{(i)})_{i\in\mathbb{Z}}$ on $[0,T]$ define

$$\|W\|^2_{[t]} = \mathbb{E}\left[\sup_{0\le\tilde{t}\le t}\sup_{i\in\mathbb{Z}}|w^{(i)}(\tilde{t})|^2\right], \quad 0\le t\le T.$$

For processes $W = (w^{(i)})_{i\in\mathbb{Z}}$ on $[-r,T]$ define

$$\|W\|^2_{t} = \mathbb{E}\left[\sup_{i\in\mathbb{Z}}\sup_{-r\le\tilde{t}\le t}|w^{(i)}(\tilde{t})|^2\right], \quad 0\le t\le T.$$

An integral representation of solutions of (3.1) is introduced in the following lemma.

Lemma 3.1. *Let $F^{(i)}, G \in C([0,T], L^2(\Omega))$ be uniformly bounded. Any bounded continous solution $v = [v_i]_{i\in\mathbb{Z}}$ of the system*

$$\begin{aligned}\frac{d}{dt}v^{(i)}(t) - \frac{v^{(i+1)}(t) - 2v^{(i)}(t) + v^{(i-1)}(t)}{h^2} &= F^{(i)}(t) + G(t)\dot{B}_t\\ \textit{for}\quad t\in[0,T],\ i\in\mathbb{Z}&\end{aligned}\tag{3.2}$$

has the representation

$$\begin{aligned}v^{(i)}(t) = {}& e^{-\frac{2}{h^2}t}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t}{h^2}\right)^n\sum_{j=0}^{n}\binom{n}{j}v^{(i-n+2j)}(0)+\\ &+\int_0^t e^{-\frac{2}{h^2}(t-s)}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t-s}{h^2}\right)^n\sum_{j=0}^{n}\binom{n}{j}F^{(i-n+2j)}(s)ds + \int_0^t G(s)dB_s.\end{aligned}\tag{3.3}$$

Lemma 3.2. *Let $F^{(i)}, G \in C([0,T], L^2(\Omega))$ be uniformly bounded. Any bounded continuous solution $v = [v_i]_{i\in\mathbb{Z}}$ of (3.2) satisfies the estimates*

$$\|v\|^2_{[t]} \le 3\|v\|^2_{[0]} + 3t\int_0^t \|F\|^2_{[s]}ds + 12\int_0^t\|G\|^2_{[s]}ds.$$

Proof. From Lemma 3.1 we get

$$\begin{aligned}\|v^{(i)}\|_{[t]} \le {}& \left\|e^{-\frac{2}{h^2}t}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t}{h^2}\right)^n\sum_{j=0}^{n}\binom{n}{j}v^{(i-n+2j)}(0)\right\|_{[t]} +\\ &+\left\|\int_0^t e^{-\frac{2}{h^2}(t-s)}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t-s}{h^2}\right)^n\sum_{j=0}^{n}\binom{n}{j}F^{(i-n+2j)}(s)ds\right\|_{[t]} +\\ &+\left\|\int_0^t G(s)dB_s\right\|_{[t]} =: A_1 + A_2 + A_3.\end{aligned}$$

Note that

$$e^{-\frac{2}{h^2}t}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t}{h^2}\right)^n\sum_{j=0}^{n}\binom{n}{j}=1. \tag{3.4}$$

Then

$$\begin{aligned}
A_1^2 &= \left\| e^{-\frac{2}{h^2}t}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t}{h^2}\right)^n\sum_{j=0}^{n}\binom{n}{k}v^{(i-n+2j)}(0)\right\|_{[t]}^2 = \\
&= \sup_{i\in\mathbb{Z}}\sup_{\tilde{t}\le t}\left| e^{-\frac{2}{h^2}\tilde{t}}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\tilde{t}}{h^2}\right)^n\sum_{j=0}^{n}\binom{n}{k}v^{(i-n+2j)}(0)\right|^2 \le \\
&\le \sup_{\tilde{t}\le t}\left(\sum_{n=0}^{\infty}\sum_{j=0}^{n}e^{-\frac{2}{h^2}\tilde{t}}\frac{1}{n!}\left(\frac{\tilde{t}}{h^2}\right)^n\binom{n}{k}\sup_{l\in\mathbb{Z}}|v^{(l)}(0)|\right)^2 = \|v\|_{[0]}^2.
\end{aligned}$$

Now we estimate A_2.

$$\begin{aligned}
A_2^2(t) &= \mathbb{E}\left[\sup_{i\in\mathbb{Z}}\sup_{\tilde{t}\le t}\left|\int_0^{\tilde{t}} e^{-\frac{2}{h^2}(\tilde{t}-s)}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\tilde{t}-s}{h^2}\right)^n\sum_{j=0}^{n}\binom{n}{j}F^{(i-n+2j)}(s)ds\right|^2\right] \le \\
&\le \mathbb{E}\left[\sup_{\tilde{t}\le t}\left|\int_0^{\tilde{t}}\sup_{l\in\mathbb{Z}}F^{(l)}(s)ds\right|^2\right] \le t\int_0^t \|F\|_{[s]}^2 ds.
\end{aligned}$$

From Doob's martingale inequality and Itô isometry we get

$$A_3^2(t) = \mathbb{E}\left[\sup_{\tilde{t}\le t}\left|\int_0^{\tilde{t}} G(s)dB_s\right|^2\right] \le 4\mathbb{E}\left[\left|\int_0^{t} G(s)dB_s\right|^2\right] \le 4\int_0^t \|G\|_{[s]}^2\, ds.$$

From the fact that $(a+b+c)^2 \le 3a^2+3b^2+3c^2$ we get the expected estimation of the form

$$\|v\|_{[t]}^2 \le 3\|v\|_{[0]}^2 + 3t\int_0^t \|F\|_{[s]}^2 ds + 12\int_0^t \|G\|_{[s]}^2 ds. \qquad \square$$

Theorem 3.3 (Existence and Uniqueness)**.** *Assume that continuous functions f, g satisfy the Lipschitz conditions* (L1), (L2) *and boundedness conditions* (B1), (B2) *with some positive constants L_1, L_2, C_1, C_2 . Then there exists a unique solution to* (3.1).

Proof. To prove the existence we define the sequence $(u_m)_{m\in\mathbb{N}}$, where $u_m = (u_m^{(i)})_{i\in\mathbb{Z}}$ with the initial process

$$\begin{aligned}
u_0^{(i)}(t) &= \varphi^{(i)}(t) \quad \text{for} \quad t\in[-r,0], \quad i\in\mathbb{Z}, \\
u_0^{(i)}(t) &= \varphi(0) \quad\ \ \text{for} \quad t\in[0,T], \quad i\in\mathbb{Z}
\end{aligned}$$

and the recurrence SDE

$$\frac{d}{dt}u_{m+1}^{(i)}(t)-\frac{u_{m+1}^{(i+1)}(t)-2u_{m+1}^{(i)}(t)+u_{m+1}^{(i-1)}(t)}{h^2}= \\ =f(t,h\cdot i,(J_hu_m)_{(t,h\cdot i)})+g(t,(J_hu_m)_{(t,0)})\dot{B}_t \quad \text{for} \quad t\in[0,T],\ i\in\mathbb{Z}$$

with the initial condition

$$u_{m+1}^{(i)}(t)=\varphi^{(i)}(t) \quad \text{for} \quad t\in[-r,0],\ i\in\mathbb{Z}.$$

We will show that (u_m) is a Cauchy sequence in the Banach space. Let $\Delta u_m^{(i)}(t)=u_{m+1}^{(i)}(t)-u_m^{(i)}(t)$. For $m=0$ we have

$$\frac{d}{dt}\Delta u_0^{(i)}(t)-\frac{\Delta u_0^{(i+1)}(t)-2\Delta u_0^{(i)}(t)+\Delta u_0^{(i-1)}(t)}{h^2}= \\ =f(t,h\cdot i,(J_hu_0)_{(t,h\cdot i)})+g(t,(J_hu_0)_{(t,0)})\dot{B}_t \\ \text{for} \quad t\in[0,T], \quad i\in\mathbb{Z}.$$

From Lemma 3.2 we get

$$\|\Delta u_0\|_t^2\leq 3\int_0^t\|(f_0)_h\|_{[s]}^2ds+12\int_0^t\|g_0\|_{[s]}^2ds,$$

where

$$(f_0)_h=f(t,h\cdot l,J_h(u_0)_{(t,h\cdot l)})$$

and

$$g_0=g(t,J_h(u_0)_{(t,0)}).$$

The estimation of $(f_0)_h$ is of the form

$$\|(f_0)_h\|_{[s]}^2=\mathbb{E}\left[\sup_{l\in\mathbb{Z},\,\tilde{s}\leq s}|f(\tilde{s},h\cdot l,(J_hu_{(\tilde{s},h\cdot l)})_0)|^2\right]= \\ =\mathbb{E}\left[\sup_{l\in\mathbb{Z},\,\tilde{s}\leq s}|f(\tilde{s},h\cdot l,(J_hu_{(\tilde{s},h\cdot l)})_0)-f(\tilde{s},h\cdot l,0)+f(\tilde{s},h\cdot l,0)|^2\right]\leq \\ \leq 2\left[\mathbb{E}\Big[\sup_{l\in\mathbb{Z},\,\tilde{s}\leq s}|f(\tilde{s},h\cdot l,(J_hu_{(\tilde{s},h\cdot l)})_0)-f(\tilde{s},h\cdot l,0)|^2+|f(\tilde{s},h\cdot l,0)|^2\Big]\right].$$

From the Lipschitz and linear growth assumptions we get

$$\|(f_0)_h\|_{[s]}^2\leq 2\,\mathbb{E}\left[\sup_{l\in\mathbb{Z},\,\tilde{s}\leq s}\left(L_1^2\cdot|\sup_{v\in[-r,s]}((J_hu)_{(v,h\cdot l)})_0|^2+C_1^2\right)\right]\leq 2L_1^2\|u_0\|_s^2+2C_1^2.$$

The estimation of g_0 is of the form

$$\begin{aligned}
\|g_0\|^2_{[s]} &= \mathbb{E}\left[\sup_{\tilde{s}\le s}|g(\tilde{s},(J_h(u_0))_{(\tilde{s},0)})|^2\right] = \\
&= \mathbb{E}\left[\sup_{\tilde{s}\le s}|g(\tilde{s},(J_h(u_0))_{(\tilde{s},0)}) - g(\tilde{s},0) + g(\tilde{s},0)|^2\right] \le \\
&\le 2\mathbb{E}\left[\sup_{\tilde{s}\le s}|g(\tilde{s},(J_h(u_0))_{(\tilde{s},0)}) - g(\tilde{s},0)|^2\right] + 2\mathbb{E}\left[\sup_{\tilde{s}\le s}|g(\tilde{s},0)|^2\right] \le \\
&\le 2\,L_2\|u_0\|^2_s + 2\,C_2^2.
\end{aligned} \tag{3.5}$$

Hence

$$\|\Delta u_0\|_t \le \int_0^t (2L_1^2\|u_0\|^2_s + 2C_1^2)^{\frac{1}{2}}\,ds + 2\int_0^t (2\,L_2\|u_0\|^2_s + 2\,C_2^2)^{\frac{1}{2}}\,ds.$$

We now make the inductive assumption for any $m \ge 0$ and prove it for $m+1$.

$$\begin{aligned}
&\frac{d}{dt}\Delta u^{(i)}_{m+1}(t) - \frac{\Delta u^{(i+1)}_{m+1}(t) - 2\Delta u^{(i)}_{m+1}(t) + \Delta u^{(i-1)}_{m+1}(t)}{h^2} = \\
&= f(t,h\cdot i,(J_h(u_m))_{(t,h\cdot i)}) - f(t,h\cdot i,(J_h(u_{m-1}))_{(t,h\cdot i)}) + \\
&\quad + g(t,(J_h(u_m))_{(t,0)}) - g(t,(J_h(u_{m-1}))_{(t,0)}) \quad \text{for} \quad t\in[0,T],\ i\in\mathbb{Z}
\end{aligned}$$

From Lemma 3.2 we get the estimation

$$\|\Delta u_{m+1}\|^2_t \le 3\int_0^t \|(\Delta f_m)_h\|^2_{[s]} + 12\int_0^t \|\Delta g_m\|^2_{[s]}ds,$$

where

$$(\Delta f_m)_h = f(t,h\cdot i,(J_h(u_m))_{(t,h\cdot i)}) - f(t,h\cdot i,(J_h(u_{m-1}))_{(t,h\cdot i)})$$

and

$$\Delta g_m = g(t,(J_h(u_m))_{(t,0)}) - g(t,(J_h(u_{m-1}))_{(t,0)}).$$

We estimate $(\Delta f_m)_h$ and $(\Delta g_m)_h$. By the Lipschitz conditions (L1) and (L2), we get the estimations

$$\begin{aligned}
\|(\Delta f_m)_h\|_{[s]} &= \|f(s,h\cdot i,(J_h(u_m))_{(s,h\cdot i)}) - f(s,h\cdot i,(J_h(u_{m-1}))_{(s,h\cdot i)})\|_{[s]} \le \\
&\le L_1\mathbb{E}\sup i\in\mathbb{Z}\sup_{-r\le\tilde{s}\le s}|(J_h(u_m))_{(\tilde{s},h\cdot i)} - (J_h(u_{m-1}))_{(\tilde{s},h\cdot i)}| \le \\
&\le L_1\|\Delta u_m\|_s.
\end{aligned}$$

Analogously we prove that

$$\|(\Delta g_m)\|_{[s]} \le L_2\|\Delta u_m\|_{[s]},$$

consequently

$$\|\Delta u_m\|_t^2 \leq const. \frac{\left(3tL_1^2 + 12L_2^2\right)^m}{m!}.$$

It follows that the partial sums

$$u_k^{(i)} = u_0^{(i)} + \sum_{m=0}^{k-1} \Delta u_m^{(i)}$$

are uniformly convergent on $[0, T]$. Denote the limit by $u^{(i)}(t)$. Then $u^{(i)}(t)$ is a continuous process. Thus $u^{(i)}(t)$ is a solution of (3.1) on $[0, T]$.

The proof of uniqueness is analogous to the one in [6]. □

Now we define the stability of MOL (3.1). Let processes $\phi_1^{(i)}, \phi_2 \in C([0, T], L^2(\Omega))$ be adapted and bounded and let $\Delta\varphi$ be deterministic on $[-r, 0]$. The perturbed system related to (3.1) is of the form

$$\begin{cases} \dfrac{d}{dt}\bar{u}^{(i)}(t) - \dfrac{\bar{u}^{(i+1)}(t) - 2\bar{u}^{(i)}(t) + \bar{u}^{(i-1)}(t)}{h^2} = \\ = f(t, h \cdot i, (J_h\bar{u})_{(t,h\cdot i)}) + g(t, (J_h\bar{u})_{(t,0)})\dot{B}_t + \\ \quad + \phi_1^{(i)}(t) + \phi_2(t)\dot{B}_t & \text{for} \quad t \in [0, T], \ i \in \mathbb{Z}, \\ \bar{u}^{(i)}(t) = \varphi^{(i)}(t) + \Delta\varphi^{(i)}(t) & \text{for} \quad i \in \mathbb{Z}, \ t \in [-r, 0], \end{cases} \tag{3.6}$$

where $\phi_1^{(i)}$ and ϕ_2 are the perturbations of the RHS of (3.1) and $\Delta\varphi^{(i)}(t)$ is the perturbation of the initial conditions. We say that the method of lines (3.1) is stable if and only if

$$\|\bar{u} - u\|_t \to 0 \quad \text{as} \quad h \to 0$$

provided that

$$\begin{aligned} \|\phi_1\|_t \leq \epsilon_1 \to 0 \quad &\text{as} \quad h \to 0, \\ \|\phi_2\|_t \leq \epsilon_2 \to 0 \quad &\text{as} \quad h \to 0, \\ \|\Delta\varphi\|_0 \leq \epsilon_3 \to 0 \quad &\text{as} \quad h \to 0 \end{aligned}$$

on $[0, T]$. In the following theorem we show the stability of the method of lines (3.1).

Theorem 3.4. *Assume that the functions f, g satisfy the assumptions of Theorem 3.3. Then the method of lines* (3.1) *is stable.*

Proof. Let u be the solution of (3.1) and $\bar{u}$ be a solution of (3.6). Let $z^{(i)}(t) = \bar{u}^{(i)}(t) - u^{(i)}(t)$. Then, from Lemma 3.1, $z^{(i)}$ is of the form

$$z^{(i)}(t) = e^{-\frac{2}{h^2}t} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{t}{h^2}\right)^n \sum_{j=0}^{n} \binom{n}{j} \Delta\varphi^{(i-n+2j)}(0) +$$

$$+ \int_0^t e^{-\frac{2}{h^2}(t-s)} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{t-s}{h^2}\right)^n \sum_{j=0}^{n} \binom{n}{j} \tilde{f}_h^{(i-n+2j)}(s) ds + \int_0^t \tilde{g}_h(s) dB_s,$$

where
$$\tilde{f}_h^{(i)}(t) = f(t, h \cdot i, (J_h \bar{u})_{(t,h\cdot i)}) - f(t, h \cdot i, (J_h u)_{(t,h\cdot i)}) + \phi_1^{(i)}(t)$$
and
$$\tilde{g}_h(s) = g(t, (J_h \bar{u})_{(t,0)}) - g(t, (J_h u)_{(t,0)}) + \phi_2^{(i)}(s).$$
From Lemma 3.2 we get the estimation of the form

$$\|z\|2_t \quad \leq 3\|\Delta\varphi\|_0^2 + 3t \int\limits_0^t \|\tilde{f}_h\|_{[s]}^2 ds + 12 \int\limits_0^t \|\tilde{g}_h\|_{[s]}^2 ds.$$

We estimate $\tilde{f}_h$. By the condition (L1), we get the estimation

$$\|\tilde{f}_h\|_{[s]} \leq \left[2L_1^2 \cdot \|z\|_s^2 + 2\epsilon_1^2\right]^{\frac{1}{2}}.$$

Clearly,

$$\begin{aligned}\|\tilde{f}_h\|_{[s]}^2 = \mathbb{E}\left[\sup_{i\in\mathbb{Z},\tilde{s}\leq s} |f(\tilde{s}, h \cdot i, (J_h\bar{u})_{(\tilde{s},h\cdot i)}) - f(\tilde{s}, h \cdot i, (J_h u)_{(\tilde{s},h\cdot i)}) + \phi_1^{(i)}(\tilde{s})|^2\right] \leq \\ \leq 2\mathbb{E}\left[\sup_{i\in\mathbb{Z},\tilde{s}\leq s} |f(\tilde{s}, h \cdot i, (J_h\bar{u})_{(\tilde{s},h\cdot i)}) - f(\tilde{s}, h \cdot i, (J_h u)_{(\tilde{s},h\cdot i)})|^2\right] + \\ + 2\mathbb{E}\left[\sup_{i\in\mathbb{Z},\tilde{s}\leq s} |\phi_1^{(i)}(\tilde{s})|^2\right] \leq 2L_1^2 \cdot \|z\|_s^2 + 2\epsilon_1^2.\end{aligned}$$

Now we estimate $\tilde{g}_h$. By the condition (L2), we get

$$\|\tilde{g}_h\|_{[s]} \leq \left[2L_2^2 \cdot \|z\|_s^2 + 2\epsilon_2^2\right]^{\frac{1}{2}}.$$

Clearly,

$$\begin{aligned}\|\tilde{g}_h\|_{[s]}^2 = \mathbb{E}\left[\sup_{\tilde{s}\leq s} |g(\tilde{s}, (J_h\bar{u})_{(\tilde{s},0)}) - g(\tilde{s}, (J_h u)_{(\tilde{s},0)}) + \phi_2(\tilde{s})|^2\right] \leq \\ \leq 2\mathbb{E}\left[\sup_{\tilde{s}\leq s} |g(\tilde{s}, (J_h\bar{u})_{(\tilde{s},0)}) - g(\tilde{s}, (J_h u)_{(\tilde{s},0)})|^2\right] + \\ + 2\mathbb{E}\left[\sup_{\tilde{s}\leq s} |\phi_1(\tilde{s})|^2\right] \leq 2L_2^2 \cdot \|z\|_s^2 + 2\epsilon_2^2.\end{aligned}$$

From the above estimations we get

$$\|z\|_t^2 \leq 3\epsilon_3^2 + 3t \int\limits_0^t \left(2L_1^2\|z\|_s^2 + 2\epsilon_1^2\right) ds + 12 \int\limits_0^t \left(2L_2^2\|z\|_s^2 + 2\epsilon_2^2\right) ds.$$

Hence

$$\|z\|_t^2 \leq 3\epsilon_3^2 + 6L_1^2 t \int\limits_0^t \|z\|_s^2 ds + 6\epsilon_1^2 t^2 + 24L_2^2 \int\limits_0^t \|z\|_s^2 ds + 24t\epsilon_2^2$$

and

$$\|z\|_t^2 \le 3\epsilon_3^2 + 6\epsilon_1^2 t^2 + 24t\epsilon_2^2 + \left(6L_1^2 t + 24L_2^2\right) \int_0^t \|z\|_s^2 ds.$$

From Gronwall's Lemma

$$\|z\|_t^2 \le \left(3\epsilon_3^2 + 6\epsilon_1^2 t^2 + 24t\epsilon_2^2\right) e^{t\left(6L_1^2 t + 24L_2^2\right)}$$

and for $\epsilon_1 \to 0$, $\epsilon_2 \to 0$, $\epsilon_3 \to 0$ we have $\|z\|_t \to 0$ which finishes the proof of the stability of the method of lines. $\square$

Acknowledgements
The author would like to thank the referee for very helpful suggestions which greatly improved this paper.

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Received: August 28, 2013.
Revised: November 26, 2013.
Accepted: December 18, 2013.