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# ON SPECTRAL STABILITY FOR RANK ONE SINGULAR PERTURBATIONS

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**Abstract.** We study the embedded point spectrum of rank one singular perturbations of an arbitrary self-adjoint operator $A$ on a Hilbert space $\mathcal{H}$. These perturbations can be regarded as self-adjoint extensions of a densely defined closed symmetric operator $B$ with deficiency indices $(1, 1)$. Assuming the deficiency vector of $B$ is cyclic for its self-adjoint extensions, we prove that the spectrum of $A$ contains a dense $G_\delta$ subset on which no eigenvalues occur for the rank one singular perturbations considered. We show this is equivalent to the existence of a dense $G_\delta$ set of rank one singular perturbations of $A$ such that their eigenvalues are isolated. The approach presented here unifies points of view taken by different authors.

**Keywords:** self-adjoint extension, rank one singular perturbation, embedded point spectra, singular continuous spectrum.

**Mathematics Subject Classification:** 47B02, 47B25, 47A55, 47A10.

## 1. INTRODUCTION

A fundamental problem in spectral theory is to understand the behavior of spectra of self-adjoint operators when these operators are perturbed. One of the most natural types of perturbations are rank one regular perturbations, that is, perturbations of the form

$$A_\alpha = A + \alpha\langle\varphi, \cdot\rangle\varphi,$$

where $\varphi$ is a cyclic vector for $A$ and the symbol $\langle\cdot,\cdot\rangle$ denotes inner product in $\mathcal{H}$. In particular, it is known that if $I$ is an interval contained in the spectrum of $A$, $\sigma(A)$, then it is possible for $A_\alpha$ to have dense point spectrum $\sigma_p(A_\alpha)$ in $I$ for *a.e.* $\alpha \in \mathbb{R}$ in Lebesgue sense, see [15]. However, this cannot happen for every $\alpha \in \mathbb{R}$. As shown in [6] and [9], there exists a dense $G_\delta$ set $\Omega \subset \mathbb{R}$ (a countable intersection of open sets) such that if $\alpha \in \Omega$, then $\sigma_p(A_\alpha) \cap I$ is empty and if $\alpha \in \mathbb{R} \setminus \Omega$, then there is a dense $G_\delta$ set $F \subset I$ such that $\sigma_p(A_\alpha) \cap F$ is empty. Nevertheless, in some situations the pure point spectrum is generic (see [2]). Related problems were studied in [10] for Sturm–Liouville operators with local perturbations.

 

A natural question is whether similar results hold for rank one singular perturbations given by the formal expression

$$A_\alpha = A + \alpha\langle\varphi, \cdot\rangle\varphi$$

with $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}$ where $\mathcal{H}_s \subseteq \mathcal{H} \subseteq \mathcal{H}_{-s},\ s \geq 0$ denotes the $A$-scale of Hilbert spaces which will be defined in Section 2. The symbol $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathcal{H}_{-s}$ and $\mathcal{H}_s$ or simply the action of linear functionals on $\mathcal{H}_s$. Rank one singular perturbations are operators on the underlying Hilbert space whose domains are different from the domain of the unperturbed operator and the difference of their resolvents is a rank one bounded operator. In [12] this question was considered for the case when $\varphi \in \mathcal{H}_{-1} \setminus \mathcal{H}$, i.e. for so-called form bounded singular perturbations and $A$ being semi-bounded.

The case addressed in this paper includes the more general situation when form unbounded singular perturbations, i.e. $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$ are considered. According to [1], the difference between the two cases lies in the fact that if $\varphi \in \mathcal{H}_{-1} \setminus \mathcal{H}$, the formal expression $A_\alpha$ determines a single operator, whereas if $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$, the operator associated with $A_\alpha$ is not uniquely determined. Specifically, in the case $\varphi \in \mathcal{H}_{-1} \setminus \mathcal{H}$, the form-sum method is used while when $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$ this method fails. However, rank one singular perturbations can be regarded as self-adjoint extensions of the restriction $A$ to $Ker\varphi$, the subspace of $D(A)$ where $\varphi$ vanishes. This restriction turns out to be a densely defined closed symmetric operator with deficiency indices $(1, 1)$. These extensions will be denoted by $A^\gamma$ and we will consider them as the rank one singular perturbations of $A$. The relationship between the coupling constant $\alpha$ and the extension parameter $\gamma$ will be explained in Section 2. Assuming that $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}$ and $(A - iI)^{-1}\varphi$ is cyclic for $A$, the main results in the current article are the following:

**Theorem 1.1** (Forbidden Energies)**.** *The set of points in $\sigma(A)$ which are not eigenvalues for any $A^\gamma$, with $\gamma \in \mathbb{R}$, contains a dense $G_\delta$ set in $\sigma(A)$.*

**Theorem 1.2** (Forbidden Extension Parameters)**.** *The set*

$$\{\gamma \in \mathbb{R} \mid \sigma_p(A^\gamma) \cap \sigma(A) = \varnothing\}$$

*is dense $G_\delta$ in $\mathbb{R}$.*

The term "forbidden" is motivated by the corresponding results for rank one regular perturbations of [6, 9, 10]. We call "energies" to the elements in $\sigma(A)$ and by "Forbidden Energies", we mean to the energies which are not eigenvalues for rank one singular perturbations $A^\gamma$ of $A$. The name "Forbidden Extension Parameters" is an analogy to "Forbidden Coupling Constants" which was used in [14]. The present paper essentially unifies the methods of [6] and [9] in the framework of self-adjoint extensions. Following the approach of [9] we get Theorem 1.1. On the other hand the ideas of [6] lead to Theorem 1.2 and allow to show that actually the main theorems are equivalent.

The paper is divided as follows. In Section 2 both the von Neumann's Extensions Theory and the theoretical framework given in [1] for rank one singular perturbations are provided. In Section 3 some results of [9] originally proved for Borel–Stieltjes transforms are extended for Nevanlinna–Herglotz functions. In Section 4 in order to illustrate what happens when the spectrum of self-adjoint extensions is not simple, a version of the well-known theorem from Aronszajn–Donoghue Theory on characterization of eigenvalues by improper integrals when self-adjoint extensions are reduced to a cyclicity space is shown. Then with this result Theorem 1.1 is proven. In Section 5 a proposition on forbidden extension parameters for self-adjoint extensions of a densely defined closed symmetric operator with deficiency indices $(1,1)$, denoted by $T_\theta$, is obtained and Theorem 1.2 is deduced by transforming the rank one singular perturbations $A^\gamma$ in terms of $T_\theta$ through a homeomorphism. From this theorem, we concluded both the aforementioned equivalence and a result on forbidden energies in the essential spectrum which cannot be eigenvalues of $A$. Other consequences of these results are that for a dense $G_\delta$ set either of rank one singular perturbations of $A$ or self-adjoint extensions of a symmetric operator, their eigenvalues are isolated and if we assume absolutely continuous spectrum is empty, there is pure singular continuous spectrum for this dense $G_\delta$ family of operators.

## 2. PRELIMINARIES

### 2.1. SELF-ADJOINT EXTENSIONS

We recall the von Neumann Extension Theorem for symmetric operators. For this, the following definition is given.

**Definition 2.1** ([11, Definition 2.2], [13, Equation 7.1.44])**.** Let $B$ denote a densely defined closed symmetric operator on a Hilbert space $\mathcal{H}$. We call deficiency spaces of $B$ to the sets

$$K_\pm(B) := \operatorname{Ran}(B \pm iI)^\perp = \operatorname{Ker}(B^* \mp iI),$$

where $\perp$ denotes orthogonal complement in $\mathcal{H}$ and $B^*$ is the adjoint operator to $B$. Also, we call deficiency indices of $B$ to the pair $(d_+(B), d_-(B))$, where

$$d_\pm(B) := \dim K_\pm(B).$$

Let $\mathcal{B}(B)$ denote the set of closed symmetric extensions of $B$ and $\mathcal{V}(B)$ the set of partial isometries from $K_+(B)$ to $K_-(B)$. We state the next theorem.

**Theorem 2.2** ([11, Theorem 13.9], [13, Theorem 7.4.1])**.** *Let $B$ denote a densely defined closed symmetric operator on $\mathcal{H}$. There exists a bijective mapping from $\mathcal{V}(B)$ to $\mathcal{B}(B)$ given by*

$$V \mapsto T_V := B^* \restriction_{D(T_V)},$$

*where*

$$D(T_V) = D(B) \dotplus (I + V)D(V).$$

*Furthermore, $T_V$ is self-adjoint if and only if $V$ is unitary from $K_+(B)$ to $K_-(B)$.*

Suppose that $B$ in the above theorem has deficiency indices $(1,1)$ and $u_\pm \in K_\pm(B)$ is a generating vector with norm equal to 1. The vector $u_\pm$ is called deficiency vector. For each $\theta \in [0,\pi)$ one defines the operator

$$V_\theta : K_+(B) \longrightarrow K_-(B), \quad \text{where } V_\theta(u_+) := e^{-2i\theta} u_-. \tag{2.1}$$

Denote the self-adjoint extensions of $B$ given by Theorem 2.2 as $T_\theta$, with $\theta \in [0,\pi)$, where

$$D(T_\theta) = D(B) \dotplus \operatorname{span}\left\{u_+ + e^{-2i\theta} u_-\right\} \tag{2.2}$$

and

$$T_\theta(\eta + a u_+ + a e^{-2i\theta} u_-) = B\eta + a i u_+ - a i e^{-2i\theta} u_-, \quad \eta \in D(B), a \in \mathbb{C}.$$

Denote by $\mathcal{M}$ the cyclicity space of $u_+$ for any $T_\theta$ which by definition is

$$\mathcal{M} := \overline{\operatorname{span}\{(T_\theta - zI)^{-1} u_+ : z \in \mathbb{C} \setminus \mathbb{R}\}}. \tag{2.3}$$

**Remark 2.3.** We know that $\mathcal{M}$ does not depend on $\theta$ and is a reducing subspace for $T_\theta$, for all $\theta \in [0,\pi)$ (see [3, Section 2], [5, Lemma 4.5]). Therefore, one has the restrictions $T_\theta \restriction_{\mathcal{M}}$ acting on the Hilbert space $\mathcal{M}$ with domain

$$D(T_\theta \restriction_{\mathcal{M}}) := D(T_\theta) \cap \mathcal{M}$$

which are self-adjoint operators and have simple spectrum since by definition $u_+$ is cyclic for $T_\theta \restriction_{\mathcal{M}}$.

## 2.2. RANK ONE SINGULAR PERTURBATIONS

Let $A$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Consider the $A$-scale of Hilbert spaces

$$\mathcal{H}_s \subseteq \mathcal{H} \subseteq \mathcal{H}_{-s},$$

where $\mathcal{H}_s := \left(D(|A|^{\frac{s}{2}}), \|\cdot\|_s\right)$ with $\|\eta\|_s := \|(|A|+I)^{\frac{s}{2}}\eta\|$ for all $s \geqslant 0$ and $\mathcal{H}_{-s}$ is the completion of $\mathcal{H}$ with respect to the norm $\|\cdot\|_{-s}$, i.e. the space of linear functionals with its usual norm $\left(\mathcal{H}_s^*, \|\cdot\|_{\mathcal{H}_s^*}\right)$. Given $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}$ and $\alpha \in \mathbb{R}$, rank one singular perturbations of $A$ are defined by the formal expression

$$A_\alpha = A + \alpha \langle \varphi, \cdot \rangle \varphi, \tag{2.4}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathcal{H}_{-2}$ and $\mathcal{H}_2$ or simply the action of linear functionals. We obtain self-adjoint realizations on $\mathcal{H}$ of the expression $A_\alpha$, which are self-adjoint extensions of a densely defined closed symmetric operator with deficiency indices $(1,1)$. The following result is the key.

**Lemma 2.4** ([1, Lemma 1.2.3]). *Let $A$ be a self-adjoint operator on $\mathcal{H}$ and $\varphi \in \mathcal{H}_{-2} \backslash \mathcal{H}$. Then*

$$\dot{A} := A \restriction_{D(\dot{A})}, \text{ where } D(\dot{A}) := \{\eta \in D(A) : \langle \varphi, \eta \rangle = 0\}$$

*is a densely defined closed symmetric operator with deficiency indices* $(1,1)$.

We now briefly recall the approach of [1]. Consider the operator

$$(A \pm iI)^{-1} : \mathcal{H}_{s-2} \longrightarrow \mathcal{H}_s, \ s = 0, 1 \tag{2.5}$$

in the generalized sense, i.e. for $\phi \in \mathcal{H}_{s-2}$ and $\eta \in \mathcal{H}_s$

$$\langle \phi, (A \mp iI)^{-1} \eta \rangle = \langle (A \pm iI)^{-1} \phi, \eta \rangle.$$

By the above lemma and using the first formula of von Neumann (see [16, Theorem 8.11] and [13, Theorem 7.1.11]), it turns out that

$$D(\dot{A}^*) = D(\dot{A}) \dotplus \operatorname{span}\{g_i, g_{-i}\}, \tag{2.6}$$

where

$$g_{\pm i} := (A \mp iI)^{-1} \varphi$$

are the deficiency vectors for $\dot{A}$. We get the family of self-adjoint extensions of $\dot{A}$ given by Theorem 2.2 as $A(v)$, where $v \in \mathbb{S}^1$, the set of unimodular complex numbers, such that

$$D(A(v)) = \left\{\eta + a_+ g_i + a_- g_{-i} \in D(\dot{A}^*) : a_- = -\overline{v} a_+\right\}. \tag{2.7}$$

One has that

$$A(A^2 + I)^{-1} \varphi = \frac{1}{2}\left[(A - iI)^{-1}\varphi + (A + iI)^{-1}\varphi\right] \in \mathcal{H}.$$

So we can write (2.6) of the next form

$$D(\dot{A}^*) = D(A) \dotplus \operatorname{span}\left\{A(A^2 + I)^{-1}\varphi\right\}. \tag{2.8}$$

This makes another parametrization for the self-adjoint extensions of $\dot{A}$, denoted by $A^\gamma$ with $\gamma \in \mathbb{R} \cup \{\infty\}$, where

$$D(A^\gamma) = \left\{\eta + bA(A^2 + I)^{-1}\varphi \in D(\dot{A}^*) : \langle \varphi, \eta \rangle = \gamma b\right\}. \tag{2.9}$$

The extension parameters $v$ and $\gamma$ are related by the formula

$$v = \frac{\gamma + i}{\gamma - i}. \tag{2.10}$$

In order to define rank one singular perturbations of $A$ as self-adjoint restrictions of $\dot{A}^*$, we need to extend the linear functional $\varphi$ to $D(\dot{A}^*)$. For this purpose, we make the following remarks:

(i) If $\varphi \in \mathcal{H}_{-1} \setminus \mathcal{H}$, $\langle \varphi, (A - zI)^{-1}\varphi \rangle$ exists because $(A - zI)^{-1}\varphi \in \mathcal{H}_1$.
(ii) If $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$, $\langle \varphi, (A - zI)^{-1}\varphi \rangle$ is not well-defined since $(A - zI)^{-1}\varphi \in \mathcal{H}$ but in general $(A - zI)^{-1}\varphi \notin \mathcal{H}_2$.

In case (ii) the linear functional $\varphi$ cannot be extended to the space $D(\dot{A}^*)$ given by (2.8). So we must renormalize the expression

$$c = \langle \varphi, A(A^2 + I)^{-1}\varphi \rangle$$

and by [1, Lemma 1.3.1], the only extensions $\varphi_c$ of $\varphi$ to $D(\dot{A}^*)$ are given by

$$\langle \varphi_c, \eta + bA(A^2 + I)^{-1}\varphi \rangle := \langle \varphi, \eta \rangle + bc, \quad \eta \in D(A), b \in \mathbb{C}, c \in \mathbb{R}.$$

Then to relate the coupling constant $\alpha$ with the extension parameter $\gamma$, we must involve a real constant. This leads to the following theorem.

**Theorem 2.5** ([1, Theorems 1.3.1 and 1.3.2]). *Let $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}$. Then $A_\alpha = A^\gamma$, where*

$$\gamma = -\left(\frac{1}{\alpha} + c\right), \quad c \in \mathbb{R}. \tag{2.11}$$

*If $\varphi \in \mathcal{H}_{-1} \setminus \mathcal{H}$, $c = \langle \varphi, A(A^2 + I)^{-1}\varphi \rangle$.*

From now on, we will denote as $A^\gamma$ to the rank one singular perturbations of $A$. We conclude this section showing the relationship between the extension parameters $\theta$ of (2.2) and $\gamma$ of (2.9). Consider that arg function is valued in $[0, 2\pi)$.

**Proposition 2.6.** *Let $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}$. Consider $\dot{A}$ as in Lemma 2.4 and $T_\theta$ given by (2.2) with $B = \dot{A}$. Then $A^\gamma = T_\theta$, where*

$$\theta = \frac{1}{2} \arg\left(-\frac{\gamma + i}{\gamma - i}\right). \tag{2.12}$$

*If $\varphi \in \mathcal{H}_{-1} \setminus \mathcal{H}$,*

$$\theta = \frac{1}{2} \arg\left[-\frac{1 + \alpha(\langle \varphi, A(A^2 + I)^{-1}\varphi \rangle - i)}{1 + \alpha(\langle \varphi, A(A^2 + I)^{-1}\varphi \rangle + i)}\right].$$

*Proof.* Given $\theta \in [0, \pi)$ and $v \in \mathbb{S}^1$, $T_\theta = A(v)$ if and only if

$$\theta = \frac{1}{2} \arg(-v).$$

Substituting formula (2.10) in the last expression, one obtains (2.12). For the case $\varphi \in \mathcal{H}_{-1} \setminus \mathcal{H}$ we substitute (2.11) in (2.12) with

$$c = \langle \varphi, A(A^2 + I)^{-1}\varphi \rangle. \qquad \square$$

**Remark 2.7.**

(i) $A = T_{\frac{\pi}{2}} = A(1) = A^\infty$.

(ii) We will also denote by $\mathcal{M}$ the cyclicity space of $(A-iI)^{-1}\varphi$ for the rank one singular perturbations $A^\gamma$ as in (2.3) by taking $A^\gamma$ and $(A-iI)^{-1}\varphi$ instead of $T_\theta$ and $u_+$. In the same way, $\mathcal{M}$ does not depend on $\gamma$ and is a reducing subspace for $A^\gamma$. We then have the restrictions $A^\gamma \restriction_{\mathcal{M}}$ acting on the Hilbert space $\mathcal{M}$ with domain
$$D(A^\gamma \restriction_{\mathcal{M}}) := D(A^\gamma) \cap \mathcal{M}$$
which are self-adjoint operators and have simple spectrum since by definition $(A-iI)^{-1}\varphi$ is cyclic for $A^\gamma \restriction_{\mathcal{M}}$.

## 3. SCALAR NEVANLINNA–HERGLOTZ FUNCTIONS

We begin by showing some properties of Nevanlinna–Herglotz functions. For positive Borel measures $\mu$ such that
$$\int_{\mathbb{R}} \frac{d\mu(x)}{1+x^2} < \infty \tag{3.1}$$
we define the function $F_\mu : \mathbb{C}^+ \longrightarrow \mathbb{C}^+$, where $\mathbb{C}^+$ is the complex upper half-plane, given by
$$F_\mu(z) := \int_{\mathbb{R}} \left( \frac{1}{x-z} - \frac{x}{1+x^2} \right) d\mu(x).$$

By Canonical Integral Representation of Nevanlinna–Herglotz functions (see [4, Theorem 2.2(iii)], [11, Theorem F.1]), $F_\mu$ is a Nevanlinna–Herglotz function. We next establish some properties of these functions. The following proposition is an extension of [12, Theorem 1.2(iii)]. Although this is a well-known fact, we include a proof for the reader's convenience.

**Proposition 3.1.** *Let $\mu$ be a positive Borel measure satisfying* (3.1)*. Suppose $w \in \mathbb{R}$ holds*
$$\int_{\mathbb{R}} \frac{d\mu(x)}{(x-w)^2} < \infty.$$
*Then*
$$F_\mu(w+i0) := \lim_{\varepsilon \to 0} F_\mu(w+i\varepsilon)$$
*exists and is real.*

*Proof.* Let $d\rho(x) := \frac{d\mu(x)}{1+x^2}$ be a finite measure. Therefore, there exists the function
$$J(z) := \int_{\mathbb{R}} \frac{1}{x-z} d\rho(x).$$

Then

$$\begin{aligned} F_\mu(z) &= \int\limits_{\mathbb{R}} \frac{1+zx}{x-z} d\rho(x) \\ &= \int\limits_{\mathbb{R}} \frac{zx-z^2}{x-z} d\rho(x) + \int\limits_{\mathbb{R}} \frac{1+z^2}{x-z} d\rho(x) \\ &= z\rho(\mathbb{R}) + (1+z^2)J(z). \end{aligned}$$

Furthermore,

$$\int\limits_{\mathbb{R}} \frac{d\rho(x)}{(x-w)^2} \leqslant \int\limits_{\mathbb{R}} \frac{(1+x^2)d\rho(x)}{(x-w)^2} = \int\limits_{\mathbb{R}} \frac{d\mu(x)}{(x-w)^2} < \infty. \tag{3.2}$$

By [12, Theorem 1.2(iii)],

$$J(w+i0) := \lim_{\varepsilon \to 0} J(w+i\varepsilon)$$

exists and is real. Thus, we have concluded. □

The next lemma appears in the proof of [9, Theorem 2.1] for the case of Borel–Stieltjes transforms.

**Lemma 3.2.** *Let $\mu$ be a positive Borel measure satisfying* (3.1). *Given $w \in \mathbb{R}$, the functions*

$$G_n(w) := \int\limits_{\mathbb{R}} \frac{d\mu(x)}{(x-w)^2 + \frac{1}{n^2}}$$

*are continuous and*

$$\int\limits_{\mathbb{R}} \frac{d\mu(x)}{(x-w)^2} = \lim_{n\to\infty} G_n(\lambda).$$

*Proof.* By doing some calculations we have

$$G_n(w) = nImF_\mu\left(w + i\frac{1}{n}\right).$$

Since the function on the right hand side is continuous with $n$ fixed, so is $G_n$ for all $n \in \mathbb{N}$. By Monotonous Convergence Theorem, the second holds. □

We provide the following definitions.

**Definition 3.3.**

(i) Let $\mathcal{X}$ be a metric space. A subset $U \subseteq \mathcal{X}$ is $G_\delta$ in $\mathcal{X}$ if there is a countable family $\{U_i\}_{i\in\mathbb{N}}$ of open sets in $\mathcal{X}$ such that $U = \bigcap_{i\in\mathbb{N}} U_i$.

(ii) A subset $S \subseteq \mathbb{R}$ is called a support of a Borel measure $\mu$ if $\mu(\mathbb{R} \setminus S) = 0$.

(iii) The smallest closed support of $\mu$ is called the topological support of $\mu$ and denoted by $\mathrm{supp}(\mu)$.

Due to the previous results, a generalization of [9, Theorem 2.1] for a larger class of measures is proven.

**Proposition 3.4.** *Let $\mu$ such that* (3.1) *holds. Then*

$$\left\{ w \in \operatorname{supp}(\mu) : \int_{\mathbb{R}} \frac{d\mu(x)}{(x-w)^2} = \infty \right\} \tag{3.3}$$

*is dense $G_\delta$ in* $\operatorname{supp}(\mu)$.

*Proof.* Let $d\rho(x) := \frac{d\mu(x)}{1+x^2}$ be a finite measure and

$$\Phi := \left\{ w \in \mathbb{R} : \int_{\mathbb{R}} \frac{d\rho(x)}{(x-w)^2} = \infty \right\}.$$

By (3.2),

$$\Phi \subseteq \left\{ w \in \mathbb{R} : \int_{\mathbb{R}} \frac{d\mu(x)}{(x-w)^2} = \infty \right\}.$$

Due to that $\rho$ and $\mu$ are equivalent we have

$$\Phi \cap \operatorname{supp}(\rho) \subseteq \operatorname{supp}(\mu) \cap \left\{ w \in \mathbb{R} : \int_{\mathbb{R}} \frac{d\mu(x)}{(x-w)^2} = \infty \right\}.$$

By [9, Theorem 2.1], $\Phi$ is dense in $\operatorname{supp}(\rho)$ and hence the set (3.3) is dense in $\operatorname{supp}(\mu)$.

By continuity of $G_n$ according to Lemma 3.2 and since (3.3) turns out to be

$$\bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \left\{ w \in \operatorname{supp}(\mu) : \int_{\mathbb{R}} \frac{d\mu(x)}{(x-w)^2 + \frac{1}{n^2}} > m \right\}.$$

It follows that (3.3) is $G_\delta$ in $\operatorname{supp}(\mu)$. □

We conclude with the following corollary.

**Corollary 3.5.** *Let $\mu$ such that* (3.1) *holds. Then*

$$\operatorname{supp}(\mu) \cap \left\{ w \in \mathbb{R} : \int_{\mathbb{R}} \frac{d\mu(x)}{(x-w)^2} < \infty \right\}$$

*is a countable union of closed nowhere dense sets in* $\operatorname{supp}(\mu)$.

Our goal in the following sections will be to obtain results on forbidden energies and forbidden extension parameters for self-adjoint extensions $T_\theta \restriction_{\mathcal{M}}$ and after for rank one singular perturbations $A^\gamma$.

## 4. FORBIDDEN ENERGIES

We extend [3, Theorem 4], classical in the Aronszajn–Donoghue Theory, for the case when $u_+$ is not cyclic. Let $\theta_0 \in [0, \pi)$ fixed and $\mathcal{E}^0$ be the spectral family of $T_{\theta_0} \restriction_{\mathcal{M}}$. Define the measure $\mu^0$ such that

$$d\mu^0(x) := (1 + x^2) d\langle u_+, \mathcal{E}^0(x) u_+ \rangle.$$

We denote by $\sigma_p$ the set of eigenvalues.

**Proposition 4.1.** *For each $\theta \neq \theta_0$,*

$$\sigma_p(T_\theta \restriction_{\mathcal{M}}) = \left\{ \lambda \in \mathbb{R} : \int_{\mathbb{R}} \frac{d\mu^0(x)}{(x-\lambda)^2} < \infty, F_{\mu^0}(\lambda + i0) = \cot(\theta - \theta_0) \right\}.$$

*Proof.* From [4, Section 4] one has that $B \restriction_{\mathcal{M}}$ is a densely defined closed symmetric operator with deficiency indices $(1, 1)$ on $\mathcal{M}$. Let $R_\theta$ be its self-adjoint extensions. By definition, $u_+$ is cyclic for every $R_\theta$. Let $\mathcal{E}'$ be the spectral family of $R_{\theta_0}$ and $d\mu'(x) := (1 + x^2) d\langle u_+, \mathcal{E}'(x) u_+ \rangle$. By [3, Theorem 4],

$$\sigma_p(R_\theta) = \left\{ \lambda \in \mathbb{R} : \int_{\mathbb{R}} \frac{d\mu'(x)}{(x-\lambda)^2} < \infty, F_{\mu'}(\lambda + i0) = \cot(\theta - \theta_0) \right\}.$$

We assert $R_\theta = T_\theta \restriction_{\mathcal{M}}$. Let us first note that

$$K_\pm(B) = \operatorname{Ran}(B \pm iI)^\perp \subseteq \operatorname{Ran}(B \restriction_{\mathcal{M}} \pm iI)^\perp = K_\pm(B \restriction_{\mathcal{M}}).$$

Both $B$ and $B \restriction_{\mathcal{M}}$ have deficiency indices $(1, 1)$, therefore $K_\pm(B) = K_\pm(B \restriction_{\mathcal{M}})$. Let us show that $D(B^* \restriction_{\mathcal{M}}) = D\left[(B \restriction_{\mathcal{M}})^*\right]$.

If $\eta \in D(B^* \restriction_{\mathcal{M}})$, then $\eta = f + p_+ + p_- \in \mathcal{M}$ with $f \in D(B)$ and $p_\pm \in K_\pm(B)$. By the above, $p_\pm \in K_\pm(B \restriction_{\mathcal{M}})$ and since $K_\pm(B \restriction_{\mathcal{M}}) \subseteq \mathcal{M}$ one has $f \in D(B) \cap \mathcal{M}$. So, $\eta \in D\left[(B \restriction_{\mathcal{M}})^*\right]$.

If $\eta \in D\left[(B \restriction_{\mathcal{M}})^*\right]$, then $\eta = g + q_+ + q_-$ with $g \in D(B \restriction_{\mathcal{M}}) = D(B) \cap \mathcal{M}$ and $q_\pm \in K_\pm(B \restriction_{\mathcal{M}}) \subseteq \mathcal{M}$. We conclude $\eta \in D(B^* \restriction_{\mathcal{M}})$ and hence $B^* \restriction_{\mathcal{M}} = (B \restriction_{\mathcal{M}})^*$.

Consider the unitary operators given by (2.1). We have

$$\begin{aligned} D(R_\theta) &= D(B \restriction_{\mathcal{M}}) \dotplus K_+(B \restriction_{\mathcal{M}}) \dotplus V_\theta \left[K_+(B \restriction_{\mathcal{M}})\right] \\ &= \left[D(B) \dotplus K_+(B) \dotplus V_\theta \left(K_+(B)\right)\right] \cap \mathcal{M} \\ &= D(T_\theta \restriction_{\mathcal{M}}). \end{aligned}$$

Since in particular $R_{\theta_0} = T_{\theta_0} \restriction_{\mathcal{M}}$, we have that $\mathcal{E}' = \mathcal{E}^0$. Therefore, $\mu' = \mu^0$ and $F_{\mu'} = F_{\mu^0}$. □

We obtain this corollary.

**Corollary 4.2.** *Consider $\mu^0$ as above. Then*

$$\bigcup_{\theta\in[0,\pi)\setminus\{\theta_0\}} \sigma_p(T_\theta \restriction_{\mathcal{M}}) = \left\{\lambda \in \mathbb{R} : \int_{\mathbb{R}} \frac{d\mu^0(x)}{(x-\lambda)^2} < \infty\right\}.$$

*Proof.* By the previous proposition, if $\lambda \in \sigma_p(T_\theta \restriction_{\mathcal{M}})$ for some $\theta \in [0,\pi) \setminus \{\theta_0\}$, then $\int_{\mathbb{R}} \frac{d\mu^0(x)}{(x-\lambda)^2} < \infty$. On the other hand, suppose $\int_{\mathbb{R}} \frac{d\mu^0(x)}{(x-\lambda)^2} < \infty$. By Proposition 3.1, $F_{\mu^0}(\lambda + i0) \in \mathbb{R}$. Furthermore,

$$h : [0,\pi) \setminus \{\theta_0\} \longrightarrow \mathbb{R}, \quad \text{where } h(\theta) := \cot(\theta - \theta_0)$$

is a bijection. Thus, there exists $\theta \in [0,\pi) \setminus \{\theta_0\}$ such that $F_{\mu^0}(\lambda + i0) = h(\theta)$. By Proposition 4.1, $\lambda \in \sigma_p(T_\theta \restriction_{\mathcal{M}})$. □

We immediately show the following proposition.

**Proposition 4.3.** *Let $\theta_0$ fixed. Then the set of points in $\sigma(T_{\theta_0} \restriction_{\mathcal{M}})$ which are not eigenvalues for any $T_\theta \restriction_{\mathcal{M}}$ with $\theta \neq \theta_0$ is dense $G_\delta$ in $\sigma(T_{\theta_0} \restriction_{\mathcal{M}})$.*

*Proof.* Since $\operatorname{supp}(\mu^0) = \sigma(T_{\theta_0} \restriction_{\mathcal{M}})$, by Corollary 3.5 and Corollary 4.2 the set

$$\sigma(T_{\theta_0} \restriction_{\mathcal{M}}) \cap \bigcup_{\theta\in[0,\pi)\setminus\{\theta_0\}} \sigma_p(T_\theta \restriction_{\mathcal{M}}) \tag{4.1}$$

is a countable union of closed nowhere dense sets in $\sigma(T_{\theta_0} \restriction_{\mathcal{M}})$.

We conclude by the fact that the set of points in $\sigma(T_{\theta_0} \restriction_{\mathcal{M}})$ which are not eigenvalues for any $T_\theta \restriction_{\mathcal{M}}$ with $\theta \neq \theta_0$, is the complement in $\sigma(T_{\theta_0} \restriction_{\mathcal{M}})$ of (4.1). □

This result leads to the proof of the first main theorem, which in fact holds in a more general setting .

*Proof of Theorem 1.1.* Consider the following set

$$\{\lambda \in \sigma(A \restriction_{\mathcal{M}}) : \lambda \notin \sigma_p(A^\gamma \restriction_{\mathcal{M}}),\ for\ any\ \gamma \in \mathbb{R}\} \tag{4.2}$$

By Proposition 2.6 it turns out that

$$\begin{aligned}(4.2) &= \left\{\lambda \in \sigma(T_{\frac{\pi}{2}} \restriction_{\mathcal{M}}) : \lambda \notin \sigma_p(T_\theta \restriction_{\mathcal{M}}),\ \text{for any } \gamma \in \mathbb{R},\ \text{where } \theta = f(\gamma)\right\} \\ &= \left\{\lambda \in \sigma(T_{\frac{\pi}{2}} \restriction_{\mathcal{M}}) : \lambda \notin \sigma_p(T_\theta \restriction_{\mathcal{M}}),\ \text{for any } \theta \in [0,\pi) \setminus \left\{\frac{\pi}{2}\right\}\right\},\end{aligned}$$

where

$$f(\gamma) := \frac{1}{2} \arg\left(-\frac{\gamma + i}{\gamma - i}\right).$$

By Proposition 4.3 when $\theta_0 = \frac{\pi}{2}$, the set (4.2) is dense $G_\delta$ in $\sigma(A \restriction_{\mathcal{M}})$. If $(A - iI)^{-1}\varphi$ is cyclic for $A$, that is $\mathcal{H} = \mathcal{M}$, the result follows. □

## 5. FORBIDDEN EXTENSION PARAMETERS

Let us start proving the next lemma.

**Lemma 5.1.** *Let $\theta \in [0,\pi)$ and $E \in \mathbb{R}$. If $y \in [\operatorname{Ker}(T_\theta - EI) \setminus \{0\}] \cap \mathcal{M}$, then $\langle y, u_+ \rangle \neq 0$.*

*Proof.* Suppose there is $y \in [\operatorname{Ker}(T_\theta - EI) \setminus \{0\}] \cap \mathcal{M}$ such that $\langle y, u_+ \rangle = 0$. Given $z \in \mathbb{C} \setminus \mathbb{R}$

$$(T_\theta - \overline{z}I)^{-1} y = (E - \overline{z})^{-1} y.$$

Then

$$\langle y, (T_\theta - zI)^{-1} u_+ \rangle = \langle (T_\theta - \overline{z}I)^{-1} y, u_+ \rangle = \langle (E - \overline{z})^{-1} y, u_+ \rangle = 0.$$

Since $u_+$ is cyclic for $T_\theta \restriction_{\mathcal{M}}$ when $\theta$ is fixed, one concludes $y = 0$. □

**Definition 5.2.** Let $X$ be a Banach space and $X^*$ be its dual space. The weak topology is the weakest topology in $X$ such that each functional in $X^*$ is continuous. The weak*-topology is the weakest topology in $X^*$ such that each functional in $X^{**}$ is continuous.

**Remark 5.3.** If $X$ is a Hilbert space the weak topology and weak*-topology coincide. Therefore, by Banach–Alaoglu–Bourbaki Theorem the closed balls in a Hilbert space are compact with respect to the weak topology.

Let $\tau := [0,\pi] \times \mathbb{R} \times \mathcal{M}$, where the Hilbert space $\mathcal{M}$ is endowed with the weak topology. By the last lemma, we can define the following sets:

$$\tau_M := [0,\pi] \times \mathbb{R} \times B_M \cap \mathcal{M},$$

where $B_M$ is the closed ball in $\mathcal{H}$ with center at 0 and radius $M$,

$$Q_M := \{(\theta, E, y) \in \tau_M : y \in \operatorname{Ker}(T_\theta - EI) \text{ such that } \langle y, u_+ \rangle = 1\}.$$

**Remark 5.4.** The topological space $\tau_M$ is metrizable because $B_M \cap \mathcal{M}$ is too. It is due to the separability of $\mathcal{M}$. Further $B_M \cap \mathcal{M}$ is a convex set in $\mathcal{M}$ so that is strongly, weakly and weakly sequentially closed in $\mathcal{M}$. Therefore, $\tau_M$ is a closed subspace of $\tau$.

We propose the next definition.

**Definition 5.5.** Let $A : \mathcal{H} \longrightarrow \mathcal{H}$ be an operator. It is said to be weakly closed in $\mathcal{H}$ if given $(\eta_n)_{n\in\mathbb{N}} \subseteq D(A)$ such that

$$\eta_n \longrightarrow^w \eta \in \mathcal{H} \quad \text{and} \quad A\eta_n \longrightarrow^w y \in \mathcal{H},$$

then $\eta \in D(A)$ and $A\eta = y$.

The proof of the following proposition follows the classical argument. It only relies on the continuity of the inner product with respect to weak limits.

**Proposition 5.6.** *Let $A : \mathcal{H} \longrightarrow \mathcal{H}$ be a densely defined operator. Then $A^*$ is weakly closed on $\mathcal{H}$.*

We prove the following lemma.

**Lemma 5.7.** *The set $Q_M$ is closed in $\tau_M$.*

*Proof.* Let $(\theta_n, E_n, y_n) \in Q_M$ be a sequence such that it converges to $(\theta, E, y) \in \tau_M$. We assert $(\theta, E, y) \in Q_M$. By definition for every $n \in \mathbb{N}$, $y_n \in \text{Ker}(T_{\theta_n} - E_n I) \cap \mathcal{M}$ such that $\langle y_n, u_+ \rangle = 1$. Since $y_n \longrightarrow^w y$ one has $\langle y, u_+ \rangle = 1$ and hence $y \neq 0$. Moreover,

$$B^* y_n = T_{\theta_n} y_n = E_n y_n \longrightarrow^w Ey.$$

By Proposition 5.6, $y \in D(B^*)$ and $B^* y = Ey$. That is, there exist $\eta \in D(B)$ and $a, b \in \mathbb{C}$ such that $y = \eta + au_+ + bu_-$. Then, for each $n \in \mathbb{N}$ there exist $\eta_n \in D(B)$ and $a_n \in \mathbb{C}$ such that

$$y_n = \eta_n + a_n u_+ + a_n e^{-2i\theta_n} u_- \longrightarrow^w y = \eta + au_+ + bu_-.$$

On the other hand, by using the inner product of the graph of $B^*$

$$\begin{aligned}
\langle y_n, u_+ \rangle_{B^*} := \langle y_n, u_+ \rangle + \langle B^* y_n, B^* u_+ \rangle &= \langle y_n, u_+ \rangle + \langle E_n y_n, i u_+ \rangle \\
&= \langle y_n, u_+ \rangle + iE_n \langle y_n, u_+ \rangle \\
&\longrightarrow^w \langle y, u_+ \rangle + iE \langle y, u_+ \rangle \\
&= \langle y, u_+ \rangle_{B^*}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\langle y_n, u_+ \rangle_{B^*} &= \langle \eta_n, u_+ \rangle_{B^*} + \langle a_n u_+, u_+ \rangle_{B^*} + \langle a_n e^{-2i\theta_n} u_-, u_+ \rangle_{B^*} = 2a_n \\
\langle y, u_+ \rangle_{B^*} &= \langle \eta, u_+ \rangle_{B^*} + \langle a u_+, u_+ \rangle_{B^*} + \langle b u_-, u_+ \rangle_{B^*} = 2a.
\end{aligned}$$

Therefore, $a_n \longrightarrow a$. Then

$$\eta_n + a_n u_+ + a_n e^{-2i\theta_n} u_- \longrightarrow^w \eta + au_+ + ae^{-2i\theta} u_-.$$

By uniqueness of limits $y = \eta + au_+ + ae^{-2i\theta} u_-$. Hence, $y \in \text{Ker}(T_\theta - EI) \setminus \{0\}$. Finally, $Q_M$ is closed. □

We obtain the following identity.

**Lemma 5.8.** *Let $y_j = \eta_j + a_j e^{i\theta_j} u_+ + a_j e^{-i\theta_j} u_- \in \text{Ker}(T_{\theta_j} - E_j I)$ with $j = 1, 2$. Then*

$$-4a_1 \overline{a_2} \text{sen}(\theta_1 - \theta_2) = (E_1 - E_2) \langle y_1, y_2 \rangle.$$

The following result is formulated like in [6].

**Lemma 5.9.** *Let $F \subseteq Q_M$ be a compact set. The function $W_F : F \times F \longrightarrow \mathbb{C}$ such that*

$$W_F \left( (\theta_1, E_1, y_1), (\theta_2, E_2, y_2) \right) := \langle y_1, y_2 \rangle$$

*is continuous at least at a pair $(\varepsilon_0, \varepsilon_0) \in F \times F$.*

*Proof.* Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{M}$. We define $J_m, J_F : F \longrightarrow \mathbb{R}$ with $m \in \mathbb{N}$ and $\varepsilon = (\theta, E, y)$ by

$$J_m(\varepsilon) := \sum_{n=1}^{m} |\langle y, e_n\rangle|^2$$

and

$$J_F(\varepsilon) := \|y\|^2.$$

Due to Parseval's identity, it turns out that for each $\varepsilon \in F$

$$J_F(\varepsilon) = \sum_{n=1}^{\infty} |\langle y, e_n\rangle|^2 = \lim_{m\to\infty} J_m(\varepsilon).$$

Let $\varepsilon_k = (\theta_k, E_k, y_k) \in F$ be a sequence that converges to $\varepsilon \in F$. Then, $y_k \longrightarrow^w y$. Hence,

$$\lim_{k\to\infty} J_m(\varepsilon_k) = \sum_{n=1}^{m} |\langle y, e_n\rangle|^2 = J_m(\varepsilon).$$

Therefore, $J_F$ is pointwise limit of continuous functions. By [8, Theorem 7.3], there exists $\varepsilon_0 \in F$ such that $J_F$ is continuous at $\varepsilon_0 := (\theta_0, E_0, y_0)$.

We assert that $W_F$ is continuous at $(\varepsilon_0, \varepsilon_0)$. If $\varepsilon_n = (\theta_n, E_n, y_n) \longrightarrow \varepsilon_0$ and $\varepsilon'_n = (\theta'_n, E'_n, y'_n) \longrightarrow \varepsilon_0$, then

$$y_n, y'_n \longrightarrow^w y_0$$

and by continuity of $J_F$ at $\varepsilon_0$,

$$\|y_n\|, \|y'_n\| \longrightarrow \|y_0\|.$$

So, $y_n, y'_n \longrightarrow y_0$. Thus,

$$W_F(\varepsilon_n, \varepsilon'_n) = \langle y_n, y'_n\rangle \longrightarrow \langle y_0, y_0\rangle = W_F(\varepsilon_0, \varepsilon_0).$$

Finally, $W_F$ is continuous at $(\varepsilon_0, \varepsilon_0)$. □

Let $\mathcal{P} : \tau \longrightarrow \mathbb{R}$, $\Pi : \tau \longrightarrow [0, \pi]$, $\mathcal{R} : \tau \longrightarrow [0, \pi] \times \mathbb{R}$, $p : [0, \pi] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $q : [0, \pi] \times \mathbb{R} \longrightarrow [0, \pi]$ be projections. We follow the same procedure used in the proof of [6, Proposition 2*] and [7, Proposition 2].

**Proposition 5.10.** *Let $F \subseteq Q_M$ be a compact set. Then $\mathcal{P}(F)$ is nowhere dense in $\mathbb{R}$ if and only if $\Pi(F)$ is nowhere dense in $[0, \pi]$.*

*Proof.* Suppose that there is a compact subset $F \subseteq Q_M$ such that $\mathcal{P}(F)$ is nowhere dense and $\Pi(F)$ contains a non-empty open set of $[0, \pi]$ which one denotes as $\mathcal{I}$. Consider the partially ordered set given by the collection

$$\{F' \subseteq F : F' \textit{ is compact set in } Q_M \textit{ and } \Pi(F') \supset \mathcal{I}\}. \tag{5.1}$$

We assert that (5.1) has a minimal element. Let $\{F_\alpha\}_{\alpha\in\Delta}$ be a chain in (5.1), where $\Delta$ is an arbitrary set of indices. It turns out that $\bigcap_{\alpha\in\Delta} F_\alpha$ is a compact subset of $F$

in $Q_M$. We assert $\Pi\left(\bigcap_{\alpha\in\Delta} F_\alpha\right) \supset \mathcal{I}$. For all $\alpha \in \Delta$, there exists $G_\alpha \subseteq \mathbb{R}\times\mathcal{M}$ such that $F_\alpha = \Pi(F_\alpha)\times G_\alpha$. It follows by

$$\bigcap_{\alpha\in\Delta} F_\alpha \supseteq \bigcap_{\alpha\in\Delta} \Pi(F_\alpha) \times \bigcap_{\alpha\in\Delta} G_\alpha.$$

Therefore, $\bigcap_{\alpha\in\Delta} F_\alpha$ is a lower bound of $\{F_\alpha\}_{\alpha\in\Delta}$. We conclude by Zorn's lemma. Denote the minimal set by $\widehat{F}$.

We now prove that there exists a subset of $\widehat{F}$ whose projections under $\mathcal{P}$ and $\Pi$ are homeomorphic. By Lemma 5.9, there exists $\delta > 0$ such that for all $\varepsilon, \varepsilon' \in \widehat{F}$

$$d_{\widehat{F}\times\widehat{F}}\left((\varepsilon,\varepsilon'),(\varepsilon_0,\varepsilon_0)\right) < \delta \Rightarrow \left|W_{\widehat{F}}(\varepsilon,\varepsilon') - W_{\widehat{F}}(\varepsilon_0,\varepsilon_0)\right| < \frac{\|y_0\|^2}{2}, \tag{5.2}$$

where $d_{\widehat{F}\times\widehat{F}}$ is the metric of $\widehat{F}\times\widehat{F}$.

Let $\mathcal{U}$ be the ball in $\tau_M$ defined as

$$\mathcal{U} := \left\{\varepsilon \in \tau_M : d_M\left(\varepsilon,\varepsilon_0\right) < \frac{\delta}{3}\right\},$$

where $d_M$ denotes the metric on $\tau_M$. We show $p|_{\mathcal{R}(\widehat{F}\cap\overline{\mathcal{U}})}$ and $q|_{\mathcal{R}(\widehat{F}\cap\overline{\mathcal{U}})}$ are injective, where $\overline{\mathcal{U}}$ denotes closure in $\tau_M$. That is, for all $\varepsilon_1 = (\theta_1, E_1, y_1)$, $\varepsilon_2 = (\theta_2, E_2, y_2) \in \widehat{F}\cap\overline{\mathcal{U}}$, it suffices to prove $\theta_1 = \theta_2$ if and only if $E_1 = E_2$.

(⇒) Suppose $\theta_1 = \theta_2$. By Lemma 5.8

$$(E_1 - E_2)\langle y_1, y_2\rangle = 0. \tag{5.3}$$

Since $\varepsilon_1, \varepsilon_2 \in \overline{\mathcal{U}}$,

$$d_{\widehat{F}\times\widehat{F}}\left((\varepsilon_1,\varepsilon_2),(\varepsilon_0,\varepsilon_0)\right) = d_M\left(\varepsilon_1,\varepsilon_0\right) + d_M\left(\varepsilon_2,\varepsilon_0\right) < \delta.$$

By (5.2),

$$\left|\langle y_1, y_2\rangle - \|y_0\|^2\right| = \left|W_{\widehat{F}}(\varepsilon_1,\varepsilon_2) - W_{\widehat{F}}(\varepsilon_0,\varepsilon_0)\right| < \frac{\|y_0\|^2}{2}.$$

Thus, $\langle y_1, y_2\rangle \neq 0$ implies $E_1 = E_2$.

(⇐) Suppose $E_1 = E_2$. If $\theta_1 \neq \theta_2$, then the operators $T_{\theta_1}\restriction_{\mathcal{M}}$ and $T_{\theta_2}\restriction_{\mathcal{M}}$ have an eigenvalue in common, but that contradicts to Proposition 4.1.

Next, $p|_{\mathcal{R}(\widehat{F}\cap\overline{\mathcal{U}})}$ and $q|_{\mathcal{R}(\widehat{F}\cap\overline{\mathcal{U}})}$ are homeomorphisms. Then

$$\mathcal{P}(\widehat{F}\cap\overline{\mathcal{U}}) = p\left[\mathcal{R}(\widehat{F}\cap\overline{\mathcal{U}})\right] \text{ and } \Pi(\widehat{F}\cap\overline{\mathcal{U}}) = q\left[\mathcal{R}(\widehat{F}\cap\overline{\mathcal{U}})\right]$$

are homeomorphic to $\mathcal{R}(\widehat{F}\cap\overline{\mathcal{U}})$ and hence between them.

On the other hand, $\Pi(\widehat{F}) = \Pi(\widehat{F}\cap\mathcal{U}) \cup \Pi(\widehat{F}\setminus\mathcal{U})$, where $\Pi(\widehat{F})$ contains to $\mathcal{I}$. Since $\mathcal{P}(\widehat{F}\cap\overline{\mathcal{U}}) \subseteq \mathcal{P}(\widehat{F})$, by hypothesis it is nowhere dense in $\mathbb{R}$. But $\Pi(\widehat{F}\cap\overline{\mathcal{U}})$ is nowhere dense in $[0,\pi]$ and $\Pi(\widehat{F}\cap\mathcal{U})$ inherits that property. Then $\mathcal{I}$ is not contained in $\Pi(\widehat{F}\cap\mathcal{U})$. Hence, $\mathcal{I}\subseteq\Pi(\widehat{F}\setminus\mathcal{U})$. Moreover, $\widehat{F}\setminus\mathcal{U}$ is properly contained in $\widehat{F}$ and is compact in $Q_M$. But this contradicts the minimality of $\widehat{F}$. Consequently, $\Pi(F)$ is nowhere dense in $[0,\pi]$. The other direction is analogous. □

Finally, we arrive at the following proposition.

**Proposition 5.11.** *Let $Y$ be a countable union of closed nowhere dense sets in $\mathbb{R}$. Then*

$$\{\theta\in[0,\pi)\mid\sigma_p(T_\theta\restriction_{\mathcal{M}})\cap Y\neq\varnothing\}$$

*is a countable union of closed nowhere dense sets in $[0,\pi]$.*

*Proof.* It turns out that

$$\{\theta\in[0,\pi)\mid\sigma_p(T_\theta\restriction_{\mathcal{M}})\cap Y\neq\varnothing\}=\Pi(Q\cap\mathcal{P}^{-1}(Y)),$$

where $Q:=\{(\theta,E,y)\in\tau:y\in\operatorname{Ker}(T_\theta-EI)\text{ and }\langle y,u_+\rangle=1\}$.

By hypothesis, there is a sequence $\{Y_n\}_{n\in\mathbb{N}}$ of closed nowhere dense sets in $\mathbb{R}$ such that $Y=\bigcup_{n\in\mathbb{N}}Y_n$. For all $M\in\mathbb{N}$ we define

$$Q^{(M)}:=Q_M\cap([-M,M]\times[-M,M]\times B_M\cap\mathcal{M})$$

which is closed in $\tau_M$ by Lemma 5.7. Further, due to Tychonoff Theorem and Banach–Alaoglu–Bourbaki Theorem, $Q^{(M)}$ is compact in $\tau_M$. It is true that $Q=\bigcup_{M\in\mathbb{N}}Q^{(M)}$. Then,

$$\Pi(Q\cap\mathcal{P}^{-1}(Y))=\bigcup_{M,n\in\mathbb{N}}\Pi\left(Q^{(M)}\cap\mathcal{P}^{-1}(Y_n)\right). \tag{5.4}$$

Since $Q^{(M)}\cap\mathcal{P}^{-1}(Y_n)$ is closed and contained in a compact, it inherits compactness. Furthermore, $\mathcal{P}\left(Q^{(M)}\cap\mathcal{P}^{-1}(Y_n)\right)=\mathcal{P}\left(Q^{(M)}\right)\cap Y_n$ is contained in a nowhere dense set, thus, it inherits that property. According to the last proposition, $\Pi\left(Q^{(M)}\cap\mathcal{P}^{-1}(Y_n)\right)$ is nowhere dense. In addition, it is compact by continuity of $\Pi$. But compact implies closed in $\mathbb{R}^2$. □

We shall use the following lemma.

**Lemma 5.12.** *Let $X$ be a topological space and $D,Y\subseteq X$ closed subspaces of $X$ such that $D\subseteq Y$. If $D$ is nowhere dense in $Y$, then it is too in $X$.*

*Proof.* Suppose there exist $a\in D$ and $U\subseteq X$ open in $X$ such that $a\in U\subseteq D$. Then $a\in U\cap Y$ and $U\cap Y\subseteq D\cap Y=D$. Hence, $D$ contains a non-empty open subset in $Y$ which is a contradiction. □

With the above we can prove the following result.

**Proposition 5.13.** *Let $\theta_0$ fixed. Then*

$$\{\theta\in[0,\pi]\mid\sigma_p(T_\theta\restriction_{\mathcal{M}})\cap\sigma(T_{\theta_0}\restriction_{\mathcal{M}})=\varnothing\} \tag{5.5}$$

*is dense $G_\delta$ in $[0,\pi]$.*

*Proof.* Consider the set

$$Y := \sigma(T_{\theta_0} \restriction_{\mathcal{M}}) \cap \bigcup_{\theta\in[0,\pi)\setminus\{\theta_0\}} \sigma_p(T_\theta \restriction_{\mathcal{M}}).$$

By Proposition 4.3 and Lemma 5.12, $Y$ is a countable union of closed nowhere dense sets in $\mathbb{R}$ and by Proposition 5.11 one has that

$$\{\theta \in [0,\pi] \mid \sigma_p(T_\theta \restriction_{\mathcal{M}}) \cap Y \neq \varnothing\}$$

is a countable union of closed nowhere dense sets in $[0,\pi]$. Thus,

$$N := \{\theta \in [0,\pi] \mid \sigma_p(T_\theta \restriction_{\mathcal{M}}) \cap Y = \varnothing\}$$

is dense $G_\delta$ in $[0,\pi]$. Furthermore,

$$\sigma_p(T_\theta \restriction_{\mathcal{M}}) \cap Y = \begin{cases} \sigma_p(T_\theta \restriction_{\mathcal{M}}) \cap \sigma(T_{\theta_0} \restriction_{\mathcal{M}}) & \text{if} \quad \theta \neq \theta_0, \\ \varnothing & \text{if} \quad \theta = \theta_0. \end{cases}$$

Therefore, $\theta_0 \in N$. However,

$$\theta_0 \text{ belongs to the set (5.5) if and only if } \sigma_p(T_{\theta_0} \restriction_{\mathcal{M}}) = \varnothing.$$

Then

$$N = \{\theta \in [0,\pi] \mid \sigma_p(T_\theta \restriction_{\mathcal{M}}) \cap \sigma(T_{\theta_0} \restriction_{\mathcal{M}}) = \varnothing\} \cup \{\theta_0\}.$$

*Case 1.* If $\sigma_p(T_{\theta_0} \restriction_{\mathcal{M}}) = \varnothing$, then $N$ is equal to the set (5.5).

*Case 2.* Suppose $\sigma_p(T_{\theta_0} \restriction_{\mathcal{M}}) \neq \varnothing$. We have

$$N \setminus \{\theta_0\} = \{\theta \in [0,\pi] \mid \sigma_p(T_\theta \restriction_{\mathcal{M}}) \cap \sigma(T_{\theta_0} \restriction_{\mathcal{M}}) = \varnothing\}.$$

Since $N$ is dense in $[0,\pi]$, $N \setminus \{\theta_0\}$ is too. Moreover, if $\{F_n\}_{n\in\mathbb{N}}$ is the sequence of open sets in $[0,\pi]$ such that $N = \bigcap_{n\in\mathbb{N}} F_n$, then $N \setminus \{\theta_0\} = \bigcap_{n\in\mathbb{N}} (F_n \setminus \{\theta_0\})$ and each $F_n \setminus \{\theta_0\}$ is open in $[0,\pi]$.

In conclusion,

$$\{\theta \in [0,\pi] \mid \sigma_p(T_\theta \restriction_{\mathcal{M}}) \cap \sigma(T_{\theta_0} \restriction_{\mathcal{M}}) = \varnothing\}$$

is dense $G_\delta$ in $[0,\pi]$. □

Denote by $\sigma_{ac}$ and $\sigma_{sc}$ the absolutely continuous and singular continuous spectrum respectively. We mean by int to interior in $\mathbb{R}$ of a set. Remind that $\sigma_p$ denotes the set of eigenvalues. We get the following corollaries.

**Corollary 5.14.** *Let $\theta_0$ fixed and suppose $\sigma_{ac}(T_{\theta_0} \restriction_{\mathcal{M}}) = \varnothing$. Then*

$$\{\theta \in [0,\pi] \mid \sigma(T_\theta \restriction_{\mathcal{M}}) \cap \operatorname{int} \sigma(T_{\theta_0} \restriction_{\mathcal{M}}) \subseteq \sigma_{sc}(T_\theta \restriction_{\mathcal{M}})\}$$

*is dense $G_\delta$ in $[0,\pi]$.*

*Proof.* Follows by the invariance of absolutely continuous spectrum for self-adjoint extensions and Proposition 5.13. □

**Corollary 5.15.** *For a dense $G_\delta$ set of self-adjoint extensions of a densely defined closed symmetric operator with deficiency indices* $(1,1)$*, their eigenvalues are isolated.*

*Proof.* We make use of the invariance of essential spectrum for self-adjoint extensions and Proposition 5.13. □

Finally, we can prove the second main theorem.

*Proof of Theorem 1.2.* Taking $\theta_0 = \frac{\pi}{2}$ and $B = \dot{A}$ in Proposition 5.13, one has

$$\Theta := \left\{\theta \in [0,\pi] \mid \sigma_p(T_\theta \restriction_{\mathcal{M}}) \cap \sigma(T_{\frac{\pi}{2}} \restriction_{\mathcal{M}}) = \varnothing\right\}$$

is dense $G_\delta$ in $[0,\pi]$.

Consider the function $\Psi : \mathbb{R} \longrightarrow [0,\pi) \setminus \left\{\frac{\pi}{2}\right\}$ defined as

$$\Psi(\gamma) := \frac{1}{2}\arg\left(-\frac{\gamma+i}{\gamma-i}\right)$$

which is a homeomorphism. Setting

$$\Gamma := \{\gamma \in \mathbb{R} \mid \sigma_p(A^\gamma \restriction_{\mathcal{M}}) \cap \sigma(A \restriction_{\mathcal{M}}) = \varnothing\}, \tag{5.6}$$

applying Proposition 2.6 and making $\theta := \Psi(\gamma)$,

$$\begin{aligned}\Psi(\Gamma) &= \{\Psi(\gamma) : \sigma_p(A^\gamma \restriction_{\mathcal{M}}) \cap \sigma(A \restriction_{\mathcal{M}}) = \varnothing\} \\ &= \left\{\theta \in [0,\pi) \setminus \left\{\frac{\pi}{2}\right\} : \sigma_p\left(T_\theta \restriction_{\mathcal{M}}\right) \cap \sigma\left(T_{\frac{\pi}{2}} \restriction_{\mathcal{M}}\right) = \varnothing\right\} \\ &= \Theta \setminus \left\{\frac{\pi}{2}\right\}.\end{aligned}$$

Then $\Gamma$ and $\Theta \setminus \left\{\frac{\pi}{2}\right\}$ are homeomorphic. We conclude that $\Gamma$ is dense $G_\delta$ in $\mathbb{R}$. Finally, the theorem follows by assuming that $\mathcal{M} = \mathcal{H}$. □

Denote by $\sigma_{ess}$ the essential spectrum of $A \restriction_{\mathcal{M}}$. We now consider

$$\tau^{ess} := [0,\pi] \times \sigma_{ess} \times \mathcal{M},$$

where $\mathcal{M}$ is endowed with the weak topology and by Lemma 5.1 can define the following sets:

$$\tau_M^{ess} := [0,\pi] \times \sigma_{ess} \times B_M \cap \mathcal{M},$$

where $B_M$ is the closed ball in $\mathcal{H}$ with center at 0 and radius $M$,

$$Q_M^{ess} := \{(\theta, E, y) \in \tau_M^{ess} : y \in \operatorname{Ker}(T_\theta - EI) \textit{ such that } \langle y, u_+\rangle = 1\},$$

$$Q^{ess} := \{(\theta, E, y) \in \tau^{ess} : y \in \operatorname{Ker}(T_\theta - EI) \textit{ such that } \langle y, u_+\rangle = 1\}.$$

The results previous to Proposition 5.11 hold if we take the sets $\tau^{ess}$, $\tau_M^{ess}$, $Q_M^{ess}$ and $Q^{ess}$ instead of $\tau$, $\tau_M$, $Q_M$ and $Q$. Therefore, we can conclude the following proposition.

**Proposition 5.16.** *If $Z$ is a countable union of closed nowhere dense sets in $[0,\pi]$, then*

$$\sigma_{ess} \cap \bigcup_{\theta\in Z} \sigma_p(T_\theta \restriction_{\mathcal{M}}) \tag{5.7}$$

*is a countable union of closed nowhere dense sets in $\sigma_{ess}$ (and therefore in $\sigma(T_{\theta_0} \restriction_{\mathcal{M}})$ for some $\theta_0$ fixed).*

*Proof.* By hypothesis, there is a sequence $\{Z_n\}_{n\in\mathbb{N}}$ of closed nowhere dense sets in $[0,\pi]$ such that $Z = \bigcup_{n\in\mathbb{N}} Z_n$. We define

$$Q^{ess}_{(M)} := Q^{ess}_M \cap ([-M,M]\times[-M,M]\times B_M \cap \mathcal{M}).$$

Following the same argument as in the proof of Proposition 5.11, taking

$$\mathcal{P}(Q^{ess}\cap \Pi^{-1}(Z)) = \bigcup_{M,n\in\mathbb{N}} \mathcal{P}\left(Q^{ess}_{(M)} \cap \Pi^{-1}(Z_n)\right)$$

instead of (5.4), we conclude that (5.7) is a countable union of closed nowhere dense sets in $\sigma_{ess}$ and by Lemma 5.12, (5.7) is a countable union of closed nowhere dense sets in $\sigma(T_{\theta_0} \restriction_{\mathcal{M}})$. □

From this result we have another proof of Theorem 1.1. Denote by $\sigma_{dis}$ the discrete spectrum.

*Second Proof of Theorem 1.1.* By Theorem 1.2, (5.6) is dense $G_\delta$ in $\mathbb{R}$. Then

$$Z := \left\{\theta\in[0,\pi] \mid \sigma_p(T_\theta \restriction_{\mathcal{M}}) \cap \sigma(T_{\frac{\pi}{2}} \restriction_{\mathcal{M}}) \neq \varnothing\right\} \cup \left\{\frac{\pi}{2}\right\} \tag{5.8}$$

is a countable union of closed nowhere dense sets in $[0,\pi]$. Replacing $Z$ in Proposition 5.16, we have

$$\sigma_{ess} \cap \bigcup_{\theta\in Z} \sigma_p(T_\theta \restriction_{\mathcal{M}})$$

is a countable union of closed nowhere dense sets in $\sigma(T_{\frac{\pi}{2}} \restriction_{\mathcal{M}})$ and hence its complement in $\sigma(T_{\frac{\pi}{2}} \restriction_{\mathcal{M}})$, namely

$$\sigma_{dis}(T_{\frac{\pi}{2}} \restriction_{\mathcal{M}}) \cup \left\{\lambda\in\sigma(T_{\frac{\pi}{2}} \restriction_{\mathcal{M}}) : \lambda\notin\sigma_p(T_\theta \restriction_{\mathcal{M}}),\ for\ any\ \theta\in[0,\pi)\right\}, \tag{5.9}$$

is dense $G_\delta$ in $\sigma(T_{\frac{\pi}{2}} \restriction_{\mathcal{M}})$. Then

$$\begin{aligned}(5.9) &\subseteq \left\{\lambda\in\sigma(T_{\frac{\pi}{2}} \restriction_{\mathcal{M}}) : \lambda\notin\sigma_p(T_\theta \restriction_{\mathcal{M}}),\ for\ any\ \theta\in[0,\pi)\setminus\left\{\frac{\pi}{2}\right\}\right\}\\ &= \{\lambda\in\sigma(A \restriction_{\mathcal{M}}) : \lambda\notin\sigma_p(A^\gamma \restriction_{\mathcal{M}}),\ for\ any\ \gamma\in\mathbb{R}\}.\end{aligned}$$

It follows by assuming $\mathcal{M}=\mathcal{H}$ □

**Remark 5.17.** *Theorem 1.1 if and only if Theorem 1.2.* For the first direction, since Theorem 1.1 is a particular case of Proposition 4.3 with $\theta_0 = \frac{\pi}{2}$ and $B = \dot{A}$, we repeat the proof of Theorem 1.2. The converse is just the second proof of Theorem 1.1.

We conclude the following corollaries.

**Corollary 5.18.** *The set of points in $\sigma_{ess}$ which are not eigenvalues for any $A^\gamma \restriction_{\mathcal{M}}$, with $\gamma \in \mathbb{R} \cup \{\infty\}$, is dense $G_\delta$ in $\sigma_{ess}$.*

*Proof.* Replacing (5.8) in Proposition 5.16,

$$\{\lambda \in \sigma_{ess} : \lambda \notin \sigma_p(T_\theta \restriction_{\mathcal{M}}),\ for\ any\ \theta \in [0,\pi)\} \tag{5.10}$$

is dense $G_\delta$ in $\sigma_{ess}$. □

**Remark 5.19.** Note if $\sigma_p(A \restriction_{\mathcal{M}}) = \varnothing$, Corollary 5.18 is equal to Theorem 1.1 since

$$(5.10) = (4.2) \setminus \sigma_p(A \restriction_{\mathcal{M}}).$$

**Corollary 5.20.** *The set*

$$\{\gamma \in \mathbb{R} \mid \sigma_p(A^\gamma \restriction_{\mathcal{M}}) = \sigma_{dis}(A^\gamma \restriction_{\mathcal{M}})\}$$

*is dense $G_\delta$ in $\mathbb{R}$. Also, if $\sigma_{ac}(A \restriction_{\mathcal{M}}) = \varnothing$*

$$\{\gamma \in \mathbb{R} \mid \sigma(A^\gamma \restriction_{\mathcal{M}}) \cap \operatorname{int} \sigma(A \restriction_{\mathcal{M}}) \subseteq \sigma_{sc}(A^\gamma \restriction_{\mathcal{M}})\}$$

*is dense $G_\delta$ in $\mathbb{R}$.*

*Proof.* The proof follows similar lines to Corollary 5.14 and 5.15. □

**Remark 5.21.** We conclude that just as in the case of rank one regular perturbations the absence of absolutely continuous spectrum implies the existence of singular continuous spectrum for a dense $G_\delta$ family of rank one singular perturbations.

## 6. FINAL REMARKS

In the unified approach presented here we used properties of spectral measures following [9] and Aronszajn–Donoghue Theory to show that there is a forbidden set of energies for rank one singular perturbations. By adapting the Gordon's methods of [6], we related this set to the extension parameters for such perturbations. We found that the existence of a subset of the spectrum of an unperturbed operator, which cannot contain eigenvalues of the perturbations, is equivalent to the existence of a large family of perturbations without embedded point spectrum. In future work, the unified approach presented here will be applied to the analysis of singular finite rank and supersingular perturbations.

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