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A NEW COMPOSITION THEOREM FOR S^p-WEIGHTED PSEUDO ALMOST PERIODIC FUNCTIONS AND APPLICATIONS TO SEMILINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we establish a new composition theorem for S^p-weighted pseudo almost periodic functions under weaker conditions than the Lipschitz ones currently encountered in the literatures. We apply this new composition theorem along with the Schauder's fixed point theorem to obtain new existence theorems for weighted pseudo almost periodic mild solutions to a semilinear differential equation in a Banach space.

Keywords: S^p-weighted pseudo almost periodic, weighted pseudo almost periodicity, semilinear differential equations.

Mathematics Subject Classification: 34K14, 60H10, 35B15, 34F05.

1. INTRODUCTION

In this paper, we are mainly concerned with the existence of weighted pseudo almost periodic mild solutions to the following semilinear differential equation

$$x'(t) = Ax(t) + f\left(t, Bx(t)\right), \ t \in \mathbb{R}, \tag{1.1}$$

where A generates an exponentially stable C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $\mathbb{X}$, and B is an arbitrary bounded linear operator on $\mathbb{X}$. Here f is an appropriate function to be specified later.

The concept of pseudo almost periodicity was introduced by Zhang [1–3] in the early nineties. It is a natural generalization of the classical almost periodicity in the sense of Bochner. Recently, Agarwal and Diagana [4], and Diagana [5–7] introduced the concept of weighted pseudo almost periodic functions, which generalizes the one of pseudo almost periodicity. N'Guérékata *et al.* [8] presented stepanov-like almost automorphic functions and discussed its application to monotone differential equations. Then, in [9], Diagana introduced the concept of Stepanov-like weighted pseudo almost periodicity and applied the concept to study the existence and uniqueness of weighted pseudo almost periodic solutions to (1.1). In recent years, the existence of almost periodic, pseudo almost periodic, weighted pseudo almost periodic and S^p-weighted pseudo almost periodic solutions of various differential equations have been considerably investigated (cf. for instance [10–23]) because of its significance and applications in physics, mechanics and mathematical biology.

Motivated by the works [9, 21, 24–26], we present here a new composition theorem for Stepanov-like weighted pseudo almost periodic functions. It is different from the existing ones which are based on Lipschitz conditions. Then we apply this new result to investigate the existence of weighted pseudo almost periodic solutions to the problem (1.1).

The rest of this paper is organized as follows. In Section 2, we present some basic definitions, lemmas, and preliminary results which will be used throughout this paper. In Section 3, we prove the existence of weighted pseudo almost periodic mild solutions to the problems (1.1).

2. PRELIMINARIES AND A NEW COMPOSITION THEOREM

In this section, we introduce some basic definitions, notations, lemmas and preliminary facts which will be used in the sequel. Particularly, we prove a new composition theorem for S^p-weighted pseudo almost automorphic functions (Theorem 2.1).

Throughout the paper, let $(\mathbb{X}, \|\cdot\|), (\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces. The notation $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ stands for the Banach space of bounded linear operators from $\mathbb{Y}$ into $\mathbb{X}$ endowed with the uniform operator topology, and we abbreviate to $\mathcal{L}(\mathbb{X})$ whenever $\mathbb{Y} = \mathbb{X}$. We let $C(\mathbb{R}, \mathbb{X})$ (respectively, $C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) denote the collection of all continuous functions from $\mathbb{R}$ into $\mathbb{X}$ (respectively, the collection of all jointly continuous functions $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$). Furthermore, $BC(\mathbb{R}, \mathbb{X})$ stands for the Banach space of all bounded continuous functions from $\mathbb{R}$ into $\mathbb{X}$ equipped with the sup norm $\|x\|_\infty := \sup_{t\in\mathbb{R}} \|x(t)\|$ for each $x \in BC(\mathbb{R}, \mathbb{X})$.

Let $\mathbb{U}$ denote the set of all functions $\rho : \mathbb{R} \to (0, \infty)$, which are locally integrable over $\mathbb{R}$ such that $\rho > 0$ almost everywhere. For a given $r > 0$ and for each $\rho \in \mathbb{U}$, we set $m(r, \rho) := \int_{-r}^{r} \rho(t)dt$.

Then we define the space of weights $\mathbb{U}_\infty$ is defined by

$$\mathbb{U}_\infty := \{\rho \in \mathbb{U} : \lim_{r\to\infty} m(r, \rho) = \infty\}.$$

In addition, we define $\mathbb{U}_B$ by

$$\mathbb{U}_B := \{\rho \in \mathbb{U} : \rho \text{ is bounded and } \liminf_{t\to\infty} \rho(t) > 0\}.$$

It is clear that $\mathbb{U}_B \subset \mathbb{U}_\infty \subset \mathbb{U}$, with strict inclusions.

Definition 2.1 ([27,28])**.** A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be (Bohr) almost periodic in $t \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that $\|f(t + \tau) - f(t)\| < \varepsilon$ for each $t \in \mathbb{R}$. The collection of all such functions will be denoted by $AP(\mathbb{X})$.

Definition 2.2 ([27, 28]). A function $f \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is said to be (Bohr) almost periodic in $t \in \mathbb{R}$ uniformly in $y \in \mathbb{Y}$ if for every $\varepsilon > 0$ and any compact $K \subset \mathbb{Y}$ there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that $\|f(t+\tau, y) - f(t, y)\| < \varepsilon$ for each $t \in \mathbb{R}$ and $y \in K$. The collection of all such functions will be denoted by $AP(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$.

Now for $\rho \in \mathbb{U}_\infty$, we define

$$PAP_0(\mathbb{X}, \rho) := \left\{ f \in BC(\mathbb{R}, \mathbb{X}) : \lim_{r \to \infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \|f(t)\| \rho(t) dt = 0 \right\},$$

$$PAP_0(\mathbb{Y}, \mathbb{X}, \rho) := \Big\{ f \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X}) : f(\cdot, y) \ \text{ is bounded for each } \ y \in \mathbb{Y}$$

$$\text{and } \lim_{r \to \infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \|f(t, y)\| \rho(t) dt = 0 \ \text{ uniformly in compact subset of } \ \mathbb{Y} \Big\}.$$

Definition 2.3 ([6, 7]). Let $\rho \in \mathbb{U}_\infty$. A function $f \in BC(\mathbb{R}, \mathbb{X})$ is called weighted pseudo almost periodic if it can be expressed as $f = g + \phi$, where $g \in AP(\mathbb{X})$ and $\phi \in PAP_0(\mathbb{X}, \rho)$. The collection of such functions will be denoted by $WPAP(\mathbb{X})$.

Definition 2.4 ([6, 7]). Let $\rho \in \mathbb{U}_\infty$. A function $f \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is called weighted pseudo almost periodic if it can be expressed as $f = g + \phi$, where $g \in AP(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ and $\phi \in PAP_0(\mathbb{Y}, \mathbb{X}, \rho)$. The collection of such functions will be denoted by $WPAP(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$.

Lemma 2.5 ([7]). *The space $WPAP(\mathbb{X})$ is a closed subspace of $(BC(\mathbb{R}, \mathbb{X}), \|\cdot\|_\infty)$. This yields $WPAP(\mathbb{X})$ is a Banach space.*

Definition 2.6 ([9,20]). The Bochner transform $f^b(t, s), t \in \mathbb{R}, s \in [0, 1]$, of a function $f : \mathbb{R} \to \mathbb{X}$ is defined by

$$f^b(t, s) := f(t + s).$$

Remark 2.7 ([9]). (i) A function $\varphi(t, s), t \in \mathbb{R}, s \in [0, 1]$, is the Bochner transform of a certain function f, $\varphi(t, s) = f^b(t, s)$, if and only if $\varphi(t + \tau, s - \tau) = \varphi(s, t)$ for all $t \in \mathbb{R}, s \in [0, 1]$ and $\tau \in [s - 1, s]$.

(ii) Note that if $f = h + \varphi$, then $f^b = h^b + \varphi^b$. Moreover, $(\lambda f)^b = \lambda f^b$ for each scalar λ.

Definition 2.8 ([9]). The Bochner transform $f^b(t, s, u), t \in \mathbb{R}, s \in [0, 1]$, $u \in \mathbb{X}$ of a function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is defined by

$$f^b(t, s, u) := f(t + s, u) \quad \text{for each} \quad u \in \mathbb{X}.$$

Definition 2.9 ([9]). Let $p \in [1, \infty)$. The space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to \mathbb{X}$ such that $f^b \in L^\infty(\mathbb{R}; L^p((0, 1), \mathbb{X}))$. This is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R}; L^p)} = \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p d\tau \right)^{\frac{1}{p}}.$$

Definition 2.10 ([9]). Let $\rho \in \mathbb{U}_\infty$. A function $f \in BS^p(\mathbb{X})$ is said to be Stepanov-like weighted pseudo almost periodic (or S^p-weighted pseudo almost periodic) if it can be decomposed as $f = h + \varphi$, where $h^b \in AP\left(L^p((0,1),\mathbb{X})\right)$ and $\varphi^b \in PAP_0\left(L^p((0,1),\mathbb{X}),\rho\right)$. In other words, a function $f \in L^p_{loc}(\mathbb{R},\mathbb{X})$ is said to be Stepanov-like weighted pseudo almost periodic relatively to the weight $\rho \in \mathbb{U}_\infty$, if its Bochner transform $f^b : \mathbb{R} \to L^p((0,1),\mathbb{X})$ is weighted pseudo almost periodic in the sense that there exist two functions $h, \varphi : \mathbb{R} \to \mathbb{X}$ such that $f = h + \varphi$, where $h^b \in AP\left(L^p((0,1),\mathbb{X})\right)$ and $\varphi^b \in PAP_0\left(L^p((0,1),\mathbb{X}),\rho\right)$. We denoted by $WPAPS^p(\mathbb{X})$ the set of all such functions.

Definition 2.11 ([9]). Let $\rho \in \mathbb{U}_\infty$. A function $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$, $(t,x) \to f(t,x)$ with $f(\cdot,x) \in L^p_{loc}(\mathbb{R},\mathbb{X})$ for each $x \in \mathbb{X}$, is said to be Stepanov-like weighted pseudo almost periodic (or S^p-weighted pseudo almost periodic) if it can be decomposed as $f = h + \varphi$, where $h^b \in AP\left(\mathbb{R} \times \mathbb{Y}, L^p((0,1),\mathbb{X})\right)$ and $\varphi^b \in PAP_0\left(\mathbb{Y}, L^p((0,1),\mathbb{X}),\rho\right)$. We denote by $WPAPS^p(\mathbb{R} \times \mathbb{Y},\mathbb{X})$ the set of all such functions.

Lemma 2.12 ([9]). *The space $WPAPS^p(\mathbb{X})$ is a closed subspace of $BC(\mathbb{R}, L^p((0,1),\mathbb{X}))$, relatively to the norm $\|\cdot\|_{S^p}$, and hence is a Banach space.*

Lemma 2.13. *Let $\rho \in \mathbb{U}_\infty$ and let $f \in BS^p(\mathbb{X})$. Then $f^b \in PAP_0\left(L^p\left((0,1),\mathbb{X}\right),\rho\right)$ if and only if for any $\varepsilon > 0$,*

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int\limits_{M_{r,\varepsilon}(f)} \rho(t)dt = 0, \tag{2.1}$$

where $M_{r,\varepsilon}(f) = \left\{ t \in [-r,r] : \left(\int_t^{t+1} \|f(s)\|^p ds \right)^{\frac{1}{p}} \geq \varepsilon \right\}$.

Proof. "Necessity". Suppose the contrary, that there exists $\varepsilon_0 > 0$ such that $\frac{1}{m(r,\rho)} \int_{M_{r,\varepsilon}(f)} \rho(t)dt$ does not converge to 0 as $r \to \infty$. Then there exists $\delta > 0$ such that for each n,

$$\frac{1}{m(r_n,\rho)} \int\limits_{M_{r_n,\varepsilon_0}(f)} \rho(t)dt \geq \delta$$

for some $r_n > n$. Then

$$
\begin{aligned}
&\frac{1}{m(r_n,\rho)}\int\limits_{-r_n}^{r_n}\Big(\int\limits_{t}^{t+1}\|f(s)\|^p ds\Big)^{\frac{1}{p}}\rho(t)dt = \\
&= \frac{1}{m(r_n,\rho)}\Big[\int\limits_{M_{r_n,\varepsilon_0}(f)}\Big(\int\limits_{t}^{t+1}\|f(s)\|^p ds\Big)^{\frac{1}{p}}\rho(t)dt + \\
&\quad + \int\limits_{[-r_n,r_n]\setminus M_{r_n,\varepsilon_0}(f)}\Big(\int_{t}^{t+1}\|f(s)\|^p ds\Big)^{\frac{1}{p}}\rho(t)dt\Big] \geq \\
&\geq \frac{1}{m(r_n,\rho)}\int\limits_{M_{r_n,\varepsilon_0}(f)}\Big(\int\limits_{t}^{t+1}\|f(s)\|^p ds\Big)^{\frac{1}{p}}\rho(t)dt \geq \\
&\geq \frac{\varepsilon_0}{m(r_n,\rho)}\int\limits_{M_{r_n,\varepsilon_0}(f)}\rho(t)dt \geq \\
&\geq \varepsilon_0\delta,
\end{aligned}
$$

which contradicts the fact that $f^b \in PAP_0\left(L^p\left((0,1),\mathbb{X}\right),\rho\right)$, that is,

$$
\lim_{r\to\infty}\frac{1}{m(r,\rho)}\int\limits_{-r}^{r}\Big(\int\limits_{t}^{t+1}\|f(s)\|^p ds\Big)^{\frac{1}{p}}\rho(t)dt = 0.
$$

Thus (2.1) holds.

"Sufficiency". From the statement of the lemma it is clear that $\|f\|_{S^p} < \infty$ and for any $\varepsilon > 0$, there exists $r_0 > 0$ such that for $r > r_0$,

$$
\frac{1}{m(r,\rho)}\int\limits_{M_{r,\varepsilon}(f)}\rho(t)dt < \frac{\varepsilon}{\|f\|_{S^p}}.
$$

Then

$$\frac{1}{m(r,\rho)}\int_{-r}^{r}\left(\int_{t}^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}\rho(t)dt =$$

$$= \frac{1}{m(r,\rho)}\Bigg[\int_{M_{r,\varepsilon}(f)}\left(\int_{t}^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}\rho(t)dt+$$

$$+\int_{[-r,r]\setminus M_{r,\varepsilon}(f)}\left(\int_{t}^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}\rho(t)dt\Bigg] \leq$$

$$\leq \frac{1}{m(r,\rho)}\left[\|f\|_{S^p}\int_{M_{r,\varepsilon}(f)}\rho(t)dt + \varepsilon\int_{[-r,r]\setminus M_{r,\varepsilon}(f)}\rho(t)dt\right] \leq$$

$$\leq \varepsilon + \varepsilon = 2\varepsilon$$

for $r > r_0$. So $\lim_{r\to\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\left(\int_{t}^{t+1}\|f(s)\|^p ds\right)^{\frac{1}{p}}\rho(t)dt = 0$. That is,

$$f^b \in PAP_0\left(L^p\left((0,1),\mathbb{X}\right),\rho\right).$$

The proof is complete. □

Remark 2.14. When $\rho(t) = 1$ for each $t \in \mathbb{R}$, we obtain Li's *et al.* result [25, Lemma 2.7] as corollary of previous lemma.

We now present a new composition result for weighted pseudo almost periodic functions.

Theorem 2.15. *Let $\rho \in \mathbb{U}_\infty$ and let $f : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$ be a S^p-weighted pseudo almost periodic functions. Suppose that f satisfies the following conditions:*

(i) *$f(t,\psi)$ is uniformly continuous in every bounded subset $K \subset \mathbb{Y}$ uniformly in $t \in \mathbb{R}$.*
(ii) *For every bounded subset $K \subset \mathbb{Y}$, $\{f(\cdot,\psi) : \psi \in K\}$ is bounded in $WPAPS^p(\mathbb{X})$.*

If $x \in WPAP(\mathbb{Y})$, then $f(\cdot, x(\cdot)) \in WPAPS^p(\mathbb{X})$.

Proof. Since $f \in WPAPS^p(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ and $x \in WPAP(\mathbb{Y})$, we have by definition that $f = g + \phi$ and $x = \alpha + \beta$, where $g^b \in AP(\mathbb{R}\times\mathbb{Y}, L^p((0,1),\mathbb{X}))$, $\phi^b \in PAP_0(\mathbb{Y}, L^p((0,1),\mathbb{X}),\rho)$, $\alpha \in AP(\mathbb{Y})$ and $\beta \in PAP_0(\mathbb{Y},\rho)$. So the function f can be decomposed as

$$\begin{aligned} f(t,x(t)) &= g(t,\alpha(t)) + f(t,x(t)) - g(t,\alpha(t)) = \\ &= g(t,\alpha(t)) + f(t,x(t)) - f(t,\alpha(t)) + \phi(t,\alpha(t)) = \\ &= G(t) + F(t) + \Phi(t), \end{aligned}$$

where $G(t) = g(t, \alpha(t))$, $F(t) = f(t, x(t)) - f(t, \alpha(t))$ and $\Phi(t) = \phi(t, \alpha(t))$. By a standard argument, it is easy to prove that $G^b \in AP(L^p((0,1), \mathbb{X}))$ (see, e.g., [27, Theorem 2.11]). To show that $f(\cdot, x(\cdot)) \in WPAPS^p(\mathbb{X})$, it is enough to show that $F^b, \Phi^b \in PAP_0(L^p((0,1), \mathbb{X}), \rho)$.

We first prove that $F^b \in PAP_0(L^p((0,1), \mathbb{X}), \rho)$. Since $x(\cdot)$ and $\alpha(\cdot)$ are bounded, we can choose a bounded subset $K \subset \mathbb{Y}$ such that $x(\mathbb{R}), \alpha(\mathbb{R}) \subset K$. It is easy to get from (ii) that $F \in BS^p(\mathbb{X})$. Under assumption (i), f is uniformly continuous on the bounded subset $K \subset \mathbb{Y}$ uniformly for $t \in \mathbb{R}$. So, given $\varepsilon > 0$, there exists $\delta > 0$ such that $u, v \in K$ and $\|u - v\|_{\mathbb{Y}} < \delta$ imply that $\|f(t,u) - f(t,v)\| < \varepsilon$ for all $t \in \mathbb{R}$. Then we have

$$\left(\int_t^{t+1} \|f(s,u) - f(s,v)\|^p ds \right)^{\frac{1}{p}} < \varepsilon.$$

Hence, for each $t \in \mathbb{R}$, $\|\beta(s)\|_{\mathbb{Y}} < \delta$, $s \in [t, t+1]$ implies that for all $t \in \mathbb{R}$,

$$\left(\int_t^{t+1} \|F(s)\|^p ds \right)^{\frac{1}{p}} = \left(\int_t^{t+1} \|f(s, x(s)) - f(s, \alpha(s))\|^p ds \right)^{\frac{1}{p}} < \varepsilon,$$

where $\beta(s) = x(s) - \alpha(s)$. Let $M(r, \delta, \beta) := \{t \in [-r, r+1] : \|\beta(t)\|_{\mathbb{Y}} \geq \delta\}$. So we get

$$M_{r,\varepsilon}(F) = M_{r,\varepsilon}(f(\cdot, x(\cdot)) - f(\cdot, \alpha(\cdot))) \subset M(r, \delta, \beta).$$

Since $\beta \in PAP_0(\mathbb{Y}, \rho)$, by an argument similar to the proof of Proposition 3.1 in [24], we can obtain that

$$\lim_{r \to \infty} \frac{1}{m(r, \rho)} \int_{M(r,\delta,\beta)} \rho(t) dt = 0.$$

Thus

$$\lim_{r \to \infty} \frac{1}{m(r, \rho)} \int_{M_{r,\varepsilon}(F)} \rho(t) dt = 0.$$

This show that $F^b \in PAP_0(L^p((0,1), \mathbb{X}), \rho)$ by Lemma 2.13.

Next, we show that $\Phi^b \in PAP_0(L^p((0,1), \mathbb{X}), \rho)$. Since $\alpha \in AP(\mathbb{Y})$ and $g^b \in AP(\mathbb{R} \times \mathbb{Y},\ L^p((0,1), \mathbb{X}))$, $\overline{\alpha(\mathbb{R})}$ is compact and g^b is uniformly continuous in $\mathbb{R} \times \overline{\alpha(\mathbb{R})}$. Then it follows from (i) that $\phi^b = f^b - g^b$ is uniformly continuous in $u \in \overline{\alpha(\mathbb{R})}$ uniformly in $t \in \mathbb{R}$. That is, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $u, v \in \overline{\alpha(\mathbb{R})}$ and $\|u - v\|_{\mathbb{Y}} < \delta$ imply that

$$\left(\int_t^{t+1} \|\phi(s,u) - \phi(s,v)\|^p ds \right)^{\frac{1}{p}} < \frac{\varepsilon}{8} \quad \text{for all} \quad t \in \mathbb{R}. \tag{2.2}$$

Meanwhile, one can find in $\alpha(\mathbb{R})$ a finite δ-net of $\overline{\alpha(\mathbb{R})}$. Namely, there exist finite number of points $u_k \in \alpha(\mathbb{R})$, $k = 1, 2, \cdots, m$, such that for any $v \in \overline{\alpha(\mathbb{R})}$, we have $\|v - u_k\|_{\mathbb{Y}} < \delta$ for some $1 \leq k \leq m$. Let

$$\mathcal{O}_k = \{t \in \mathbb{R} : \|\alpha(t) - u_k\|_{\mathbb{Y}} < \delta\}, \quad k = 1, 2, \cdots, m.$$

Then $\mathbb{R} = \cup_{k=1}^{m} \mathcal{O}_k$. Let

$$\mathfrak{B}_1 = \mathcal{O}_1, \quad \mathfrak{B}_k = \mathcal{O}_k \backslash \left(\cup_{i=1}^{k-1} \mathcal{O}_i \right), \quad k = 2, 3, \cdots, m.$$

Thus

$$\mathbb{R} = \cup_{k=1}^{m} \mathfrak{B}_k \ \text{ and } \mathfrak{B}_i \cap \mathfrak{B}_j = \emptyset, \quad i \neq j, \quad 1 \leq i, j \leq m. \tag{2.3}$$

Define a function $h : \mathbb{R} \to \mathbb{Y}$ by $h(t) = u_k$ for $t \in \mathfrak{B}_k$, $k = 1, 2, \cdots, m$. Then $\|\alpha(t) - h(t)\|_{\mathbb{Y}} < \delta$ for $t \in \mathbb{R}$, and it is easy to get from (2.2) and (2.3) that

$$\left(\sum_{k=1}^{m} \int\limits_{\mathfrak{B}_k \cap [t,t+1]} \|\phi\left(s, \alpha(s)\right) - \phi(s, u_k)\|^p ds \right)^{\frac{1}{p}} = \tag{2.4}$$

$$= \left(\int\limits_{t}^{t+1} \Big\| \phi\big(s, \alpha(s)\big) - \phi\big(s, h(s)\big) \Big\|^p ds \right)^{\frac{1}{p}} < \frac{\varepsilon}{8}. \tag{2.5}$$

Since $\phi^b \in PAP_0\left(\mathbb{Y}, L^p((0,1), \mathbb{X}), \rho\right)$, there exists $r_0 > 0$ such that

$$\frac{1}{m(r,\rho)} \int\limits_{-r}^{r} \left(\int\limits_{t}^{t+1} \|\phi(s, u_k)\|^p ds \right)^{\frac{1}{p}} \rho(t) dt < \frac{\varepsilon}{8m^2} \tag{2.6}$$

for all $r > r_0$ and $1 \leq k \leq m$. Now by (2.3)–(2.6), for all $r > r_0$, we have

$$\frac{1}{m(r,\rho)} \int\limits_{-r}^{r} \left(\int\limits_{t}^{t+1} \|\Phi(s)\|^p ds \right)^{\frac{1}{p}} \rho(t) dt =$$

$$= \frac{1}{m(r,\rho)} \int\limits_{-r}^{r} \left(\sum_{k=1}^{m} \int\limits_{\mathfrak{B}_k \cap [t,t+1]} \|\phi\left(s, \alpha(s)\right) - \phi(s, u_k) + \phi(s, u_k)\|^p ds \right)^{\frac{1}{p}} \rho(t) dt \leq$$

$$\leq \frac{1}{m(r,\rho)}\int_{-r}^{r}\left[2^p\sum_{k=1}^{m}\left(\int_{\mathfrak{B}_k\cap[t,t+1]}\|\phi\left(s,\alpha(s)\right)-\phi(s,u_k)\|^p ds+\right.\right.$$

$$\left.\left.+\int_{\mathfrak{B}_k\cap[t,t+1]}\|\phi(s,u_k)\|^p ds\right)\right]^{\frac{1}{p}}\rho(t)dt \leq$$

$$\leq \frac{2^{1+\frac{1}{p}}}{m(r,\rho)}\int_{-r}^{r}\left(\sum_{k=1}^{m}\int_{\mathfrak{B}_k\cap[t,t+1]}\|\phi\left(s,\alpha(s)\right)-\phi(s,u_k)\|^p ds\right)^{\frac{1}{p}}\rho(t)dt+$$

$$+\frac{2^{1+\frac{1}{p}}}{m(r,\rho)}\int_{-r}^{r}\left(\sum_{k=1}^{m}\int_{\mathfrak{B}_k\cap[t,t+1]}\|\phi(s,u_k)\|^p ds\right)^{\frac{1}{p}}\rho(t)dt <$$

$$< \frac{4}{m(r,\rho)}\int_{-r}^{r}\frac{\varepsilon}{8}\rho(t)dt+\frac{4m^{\frac{1}{p}}}{m(r,\rho)}\sum_{k=1}^{m}\int_{-r}^{r}\left(\int_{\mathfrak{B}_k\cap[t,t+1]}\|\phi(s,u_k)\|^p ds\right)^{\frac{1}{p}}\rho(t)dt <$$

$$< \frac{\varepsilon}{2}+m^{\frac{1}{p}}\frac{\varepsilon}{2m} < \varepsilon,$$

which implies that $\Phi^b \in PAP_0\left(L^p((0,1),\mathbb{X}),\rho\right)$. This completes the proof. □

Remark 2.16. We obtain Li et al.'s result [25, Theorem 2.8] as an immediate consequence of previous theorem when $\rho = 1$.

Definition 2.17. A continuous function u is called a weighted pseudo almost periodic mild solution of Eq. (1.1) on $\mathbb{R}$ if $u \in WPAP(\mathbb{X})$ and $u(t)$ satisfies

$$u\left(t\right) = T\left(t-a\right)u(a)+\int_{a}^{t}T(t-s)f\left(s,Bu(s)\right)ds$$

for $t \geq a$.

3. EXISTENCE OF WEIGHTED PSEUDO ALMOST PERIODIC SOLUTIONS

In this section, we investigate the existence of weighted pseudo almost periodic mild solutions for the problem (1.1). We first list the following basic assumptions of this paper:

(H1) The operator A is the infinitesimal generator of an exponentially stable C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $\mathbb{X}$; that is, there exist constants $M > 0$, $\delta > 0$ such that $\|T(t)\| \leq Me^{-\delta t}$ for all $t \geq 0$. Moreover, $T(t)$ is compact for $t > 0$.

(H2) The operator $B : \mathbb{X} \to \mathbb{X}$ is bounded linear operator and $BK \subset K$, where K is any bounded subset of $\mathbb{X}$.

(H3) Let $p > 1$ and $q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.
(H4) The function $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ satisfies the following conditions:

(i) f is S^p-weighted pseudo almost periodic and $f(t, \cdot)$ is uniformly continuous in every bounded subset $K \subset \mathbb{X}$ uniformly in $t \in \mathbb{R}$.

(ii) There exists $L > 0$ such that

$$M_f := \sup_{t \in \mathbb{R}, \|Bx\| \leq L} \left(\int_t^{t+1} \|f(s, Bx(s))\|^p ds \right)^{\frac{1}{p}} \leq \frac{L}{\Delta(M, q, \delta)},$$

where $\Delta(M, q, \delta) = M \sqrt[q]{\frac{e^{q\delta} - 1}{q\delta}} \sum_{k=1}^{\infty} e^{-\delta k}$.

(iii) Let $\{x_n\} \subset WPAP(\mathbb{X})$ be uniformly bounded in $\mathbb{R}$ and uniformly convergent in each compact subset of $\mathbb{R}$. Then $\{f(\cdot, x_n(\cdot))\}$ is relatively compact in $BS^p(\mathbb{X})$.

(H5) The space $WPAP(\mathbb{X})$ is translation invariant, that is, the weighted function $\rho \in \mathbb{U}_\infty$ and satisfies:

$$\limsup_{t \to \infty} \frac{\rho(t + \tau)}{\rho(t)} < \infty \quad \text{and} \quad \limsup_{r \to \infty} \frac{m(r + \tau, \rho)}{m(r, \rho)} < \infty$$

for every $\tau \in \mathbb{R}$.

Remark 3.1. Note that condition (H5) was introduced by Diagana in [5–7].

To establish our first main result, we need the following technical lemma.

Lemma 3.2. *Let $\rho \in \mathbb{U}_\infty$ and suppose that assumptions* (H1)–(H3), (H4)(i) *and* (H5) *hold. Let $x : \mathbb{R} \to \mathbb{X}$ be a weighted pseudo almost periodic function and let Λ be the nonlinear operator defined by*

$$(\Lambda x)(t) := \int_{-\infty}^{t} T(t - s) f(s, Bx(s))\, ds \quad \textit{for each} \quad t \in \mathbb{R}.$$

Then Λ is continuous and maps $WPAP(\mathbb{X})$ into itself.

Proof. We first prove that Λ maps $WPAP(\mathbb{X})$ into itself. Let $x \in WPAP(\mathbb{X})$. Since $B \in \mathcal{L}(\mathbb{X})$, then $Bx \in WPAP(\mathbb{X})$. Setting $F(t) = f(t, Bx(t))$ and using Theorem 2.15 it follows that $F \in WPAPS^p(\mathbb{X})$. Write $F = \varphi + \psi$ where $\varphi^b \in AP(L^p((0, 1), \mathbb{X}))$ and $\psi^b \in PAP_0(L^p((0, 1), \mathbb{X}), \rho)$. Hence Λx can be rewritten as

$$(\Lambda x)(t) = \Phi(t) + \Psi(t),$$

where $\Phi(t) = \int_{-\infty}^{t} T(t - s)\varphi(s) ds$ and $\Psi(t) = \int_{-\infty}^{t} T(t - s)\psi(s) ds$ for each $t \in \mathbb{R}$. It follows from [25, Lemma 3.4] that the function $\Phi(t)$ is almost periodic.

Now, we show that $\Psi(t) \in PAP_0(\mathbb{X}, \rho)$. For this, we consider

$$\Psi_k(t) := \int_{t-k}^{t-k+1} T(t-s)\psi(s)ds = \int_{k-1}^{k} T(s)\psi(t-s)ds$$

for each $t \in \mathbb{R}$ and $k = 1, 2, \cdots$. From assumption (H1) and Hölder's inequality, it follows that

$$\begin{aligned}
\|\Psi_k(t)\| &\leq M \int_{t-k}^{t-k+1} e^{-\delta(t-s)} \|\psi(s)\| ds \leq \\
&\leq M \left(\int_{t-k}^{t-k+1} e^{-q\delta(t-s)} ds \right)^{\frac{1}{q}} \left(\int_{t-k}^{t-k+1} \|\psi(s)\|^p ds \right)^{\frac{1}{p}} \leq \\
&\leq M \left(\int_{k-1}^{k} e^{-q\delta s} ds \right)^{\frac{1}{q}} \left(\int_{t-k}^{t-k+1} \|\psi(s)\|^p ds \right)^{\frac{1}{p}} \leq \\
&\leq M \sqrt[q]{\frac{e^{q\delta}-1}{q\delta}} e^{-\delta k} \left(\int_{t-k}^{t-k+1} \|\psi(s)\|^p ds \right)^{\frac{1}{p}}.
\end{aligned}$$

Then, for $r > 0$, we see that

$$\begin{aligned}
&\frac{1}{m(r,\rho)} \int_{-r}^{r} \|\Psi_k(t)\| \rho(t) dt \leq \\
&\leq M \sqrt[q]{\frac{e^{q\delta}-1}{q\delta}} e^{-\delta k} \frac{1}{m(r,\rho)} \int_{-r}^{r} \left(\int_{t-k}^{t-k+1} \|\psi(s)\|^p ds \right)^{\frac{1}{p}} \rho(t) dt.
\end{aligned}$$

Since $\psi^b \in PAP_0\left(L^p((0,1), \mathbb{X}), \rho\right)$, the above inequality leads to $\Psi_k \in PAP_0(\mathbb{X}, \rho)$ for each $k = 1, 2, \cdots$. The above inequality leads also to

$$\|\Psi_k(t)\| \leq M \sqrt[q]{\frac{e^{q\delta}-1}{q\delta}} e^{-\delta k} \|\psi\|_{S^p}.$$

Since $M \sqrt[q]{\frac{e^{q\delta}-1}{q\delta}} \sum_{k=1}^{\infty} e^{-\delta k} < \infty$, we deduce from the Weierstrass test that the series $\sum_{k=1}^{\infty} \Psi_k(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$\Psi(t) = \int_{-\infty}^{t} T(t-s)\psi(s)ds = \sum_{k=1}^{\infty} \Psi_k(t)$$

and clearly $\Psi(t) \in C(\mathbb{R}, \mathbb{X})$. Applying $\Psi_k \in PAP_0(\mathbb{X}, \rho)$ and the inequality

$$\frac{1}{m(r,\rho)} \int_{-r}^{r} \|\Psi(t)\| \rho(t) dt \le$$

$$\le \frac{1}{m(r,\rho)} \int_{-r}^{r} \left\| \Psi(t) - \sum_{k=1}^{n} \Psi_k(t) \right\| \rho(t) dt + \sum_{k=1}^{n} \frac{1}{m(r,\rho)} \int_{-r}^{r} \|\Psi_k(t)\| \rho(t) dt,$$

we deduce that the uniformly limit $\Psi(\cdot) = \sum_{k=1}^{\infty} \Psi_k(t) \in PAP_0(\mathbb{X}, \rho)$. Thus we have $\Lambda x \in WPAP(\mathbb{X})$.

Now to complete the rest of the proof, we need to show that Λ is continuous on $WPAP(\mathbb{X})$. Let $\{x_n\} \subset WPAP(\mathbb{X})$ be a sequence which converges to some $x \in WPAP(\mathbb{X})$, that is, $\|x_n - x\| \to 0$ as $n \to \infty$. We may find a bounded subset $K \subset \mathbb{X}$ such that $x_n(t), x(t) \in K$ for $t \in \mathbb{R}$, $n = 1, 2, \cdots$. By (H4)(i), for any $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in K$ and $\|x - y\| < \delta$ imply that

$$\|f(t,x) - f(t,y)\| < \frac{\delta\varepsilon}{M} \quad \text{for each} \quad t \in \mathbb{R},$$

where δ, M are given in (H1). For the above $\delta > 0$, there exists $N > 0$ such that $\|Bx_n(t) - Bx(t)\| < \delta$ for all $n > N$ and all $t \in \mathbb{R}$. Therefore,

$$\|f(t, Bx_n(t)) - f(t, Bx(t))\| < \frac{\delta\varepsilon}{M}$$

for all $n > N$ and all $t \in \mathbb{R}$. Then by the Lebesgue's Dominated Convergence theorem, we have

$$\begin{aligned}
\|(\Lambda x_n)(t) - (\Lambda x)(t)\| &= \left\| \int_{-\infty}^{t} T(t-s)[f(s, Bx_n(s)) - f(s, Bx(s))] ds \right\| \le \\
&\le M \int_{-\infty}^{t} e^{-\delta(t-s)} \|f(s, Bx_n(s)) - f(s, Bx(s))\| ds < \\
&< M \int_{-\infty}^{t} e^{-\delta(t-s)} \frac{\delta\varepsilon}{M} ds \le \varepsilon
\end{aligned}$$

for all $n > N$ and all $t \in \mathbb{R}$. This implies that Λ is continuous. The proof is completed. □

Now, we are ready to state our first main result.

Theorem 3.3. *Assume the conditions* (H1)–(H5) *are satisfied, then the problem* (1.1) *has a weighted pseudo almost periodic mild solution on* $\mathbb{R}$*.*

Proof. We define the nonlinear operator $\Gamma : WPAP(\mathbb{X}) \to C(\mathbb{R},\mathbb{X})$ by

$$(\Gamma x)(t) = \int_{-\infty}^{t} T(t-s) f\left(s, Bx(s)\right) ds, \quad t \in \mathbb{R}.$$

From the statement of the above lemma it is clear that the nonlinear operator Γ is well defined and continuous. Moreover, from Lemma 3.2 we infer that $\Gamma x \in WPAP(\mathbb{X})$, that is, Γ maps $WPAP(\mathbb{X})$ into itself. For the sake of convenience, we break the proof into several steps.
Step 1. Let $\mathbb{B} = \{x \in WPAP(\mathbb{X}) : \|x\|_\infty \leq L\}$. Then $\mathbb{B}$ is a closed convex subset of $WPAP(\mathbb{X})$. We claim that $\Gamma\mathbb{B} \subset \mathbb{B}$. In fact, for $x \in \mathbb{B}$ and $t \in \mathbb{R}$, we get

$$\|(\Gamma x)(t)\| \leq \sum_{k=1}^{\infty} \left\| \int_{t-k}^{t-k+1} T(t-s) f\left(s, Bx(s)\right) ds \right\| \leq$$

$$\leq \sum_{k=1}^{\infty} M \int_{t-k}^{t-k+1} e^{-\delta(t-s)} \|f\left(s, Bx(s)\right)\| ds \leq$$

$$\leq \sum_{k=1}^{\infty} M \left(\int_{t-k}^{t-k+1} e^{-q\delta(t-s)} ds \right)^{\frac{1}{q}} \left(\int_{t-k}^{t-k+1} \|f\left(s, Bx(s)\right)\|^p ds \right)^{\frac{1}{p}} \leq$$

$$\leq \sum_{k=1}^{\infty} M \sqrt[q]{\frac{e^{q\delta}-1}{q\delta}} e^{-\delta k} M_f \leq L,$$

which implies that $\|\Gamma x\|_\infty \leq L$. Thus $\Gamma\mathbb{B} \subset \mathbb{B}$.

Step 2. Next we prove that the operator Γ is completely continuous on $\mathbb{B}$. It surfies to prove that the following statements are true.
(i) $V(t) = \{(\Gamma x)(t) : x \in \mathbb{B}$ is relatively compact in $\mathbb{X}$ for each $t \in \mathbb{R}$.
(ii) $\{\Gamma x : x \in \mathbb{B} \subset WPAP(\mathbb{X})$ is a family of equicontinuous functions.

First we show that (i) holds. Let $0 < \varepsilon < 1$ be given. For each $t \in \mathbb{R}$ and $x \in \mathbb{B}$, we define

$$(\Gamma_\varepsilon x)(t) = \int_{-\infty}^{t-\varepsilon} T(t-s) f(s, Bx(s))\, ds =$$

$$= T(\varepsilon) \int_{-\infty}^{t-\varepsilon} T(t-\varepsilon-s) f(s, Bx(s))\, ds =$$

$$= T(\varepsilon)[(\Gamma x)(t-\varepsilon)].$$

Since $T(t)(t > 0)$ is compact, then the set $V_\varepsilon(t) := \{(\Gamma_\varepsilon x)(t) : x \in \mathbb{B}\}$ is relatively compact in $\mathbb{X}$ for each $t \in \mathbb{R}$. Moreover, for each $x \in \mathbb{B}$, we get

$$\begin{aligned}
\|(\Gamma x)(t) - (\Gamma_\varepsilon x)(t)\| &= \left\| \int_{t-\varepsilon}^{t} T(t-s) f(s, Bx(s))\, ds \right\| \leq \\
&\leq M \int_{t-\varepsilon}^{t} e^{-\delta(t-s)} \|f(s, Bx(s))\| ds \leq \\
&\leq M \Big(\int_{t-\varepsilon}^{t} e^{-q\delta(t-s)} ds \Big)^{\frac{1}{q}} \Big(\int_{t-\varepsilon}^{t} \|f(s, Bx(s))\|^p ds \Big)^{\frac{1}{p}} \leq \\
&\leq M M_f \Big(\int_{t-\varepsilon}^{t} e^{-q\delta(t-s)} ds \Big)^{\frac{1}{q}}.
\end{aligned}$$

Therefore, letting $\varepsilon \to 0$, it follows that there are relatively compact sets $V_\varepsilon(t)$ arbitrarily close to $V(t)$ and hence $V(t)$ is also relatively compact in $\mathbb{X}$ for each $t \in \mathbb{R}$.

Next we prove that (ii) holds. Let $\varepsilon > 0$ be small enough and $-\infty < t_1 < t_2 < \infty$. Since $\{T(t)\}_{t \geq 0}$ is a C_0-semigroup and $T(t)$ is compact for $t > 0$, there exists $\omega = \omega(\varepsilon) < \tau_\varepsilon$ such that $t_2 - t_1 < \omega$ implies that

$$\Big\| T(\frac{t}{2}) - T\Big(\frac{t}{2} + t_2 - t_1\Big) \Big\| < \frac{\varepsilon}{\mathfrak{L}} \quad \text{for each} \quad t > 0,$$

where $\mathfrak{L} = 3MM_f \sqrt[q]{\frac{2(e^{\frac{q\delta}{2}}-1)}{q\delta}} \sum_{k=1}^{\infty} e^{-\frac{\delta(\tau_\varepsilon + k)}{2}}$.

Indeed, for $x \in \mathbb{B}$ and $t_2 - t_1 < \omega$, we have

$$\begin{aligned}
\|(\Gamma x)(t_2) - (\Gamma x)(t_1)\| \le & \left\| \int_{-\infty}^{t_1-\tau_\varepsilon} [T(t_2-s) - T(t_1-s)] f(s, Bx(s))\, ds \right\| + \\
& + \left\| \int_{t_1-\tau_\varepsilon}^{t_1} [T(t_2-s) - T(t_1-s)] f(s, Bx(s))\, ds \right\| + \\
& + \left\| \int_{t_1}^{t_2} T(t_2-s) f(s, Bx(s))\, ds \right\| \le \\
\le & \left\| \int_{\tau_\varepsilon}^{\infty} [T(t_2-t_1+s) - T(s)] f(t_1-s, Bx(t_1-s))\, ds \right\| + \\
& + M \int_{t_1-\tau_\varepsilon}^{t_1} [e^{-\delta(t_2-s)} + e^{-\delta(t_1-s)}] \| f(s, Bx(s)) \| ds + \\
& + M \int_{t_1}^{t_2} e^{-\delta(t_2-s)} \| f\big(s, Bx(s)\big) \| ds \le \\
\le & \left\| \int_{\tau_\varepsilon}^{\infty} [T(t_2-t_1+\frac{s}{2}) - T(\frac{s}{2})] f(t_1-s, Bx(t_1-s))\, ds \right\| + \\
& + M \Big(\int_{t_1-\tau_\varepsilon}^{t_1} [e^{-\delta(t_2-s)} + e^{-\delta(t_1-s)}]^q ds \Big)^{\frac{1}{q}} \Big(\int_{t_1-\tau_\varepsilon}^{t_1} \| f(s, Bx(s)) \|^p ds \Big)^{\frac{1}{p}} + \\
& + M \Big(\int_{t_1}^{t_2} e^{-q\delta(t_2-s)} ds \Big)^{\frac{1}{q}} \Big(\int_{t_1}^{t_2} \| f(s, Bx(s)) \|^p ds \Big)^{\frac{1}{p}} \le \\
\le & \frac{\varepsilon}{\mathfrak{L}} M \int_{\tau_\varepsilon}^{\infty} e^{-\frac{\delta s}{2}} \| f(t_1-s, Bx(t_1-s)) \| ds + 2M\tau_\varepsilon^{\frac{1}{q}} M_f + M\omega^{\frac{1}{q}} M_f \le \\
\le & \frac{\varepsilon}{\mathfrak{L}} M \sum_{k=1}^{\infty} \int_{\tau_\varepsilon+k-1}^{\tau_\varepsilon+k} e^{-\frac{\delta s}{2}} \| f(t_1-s, Bx(t_1-s)) \| ds + \\
& + 2M \left[\Big(\frac{\varepsilon}{6MM_f} \Big)^q \right]^{\frac{1}{q}} M_f + M\tau_\varepsilon^{\frac{1}{q}} M_f \le \\
\le & \frac{\varepsilon}{\mathfrak{L}} M \sum_{k=1}^{\infty} \Big(\int_{\tau_\varepsilon+k-1}^{\tau_\varepsilon+k} e^{-\frac{q\delta s}{2}} ds \Big)^{\frac{1}{q}} \Big(\int_{\tau_\varepsilon+k-1}^{\tau_\varepsilon+k} \| f\Big(t_1-s, Bx(t_1-s)\Big) \|^p ds \Big)^{\frac{1}{p}} + \\
& + \frac{\varepsilon}{3} + \frac{\varepsilon}{6} \le \\
\le & \frac{\varepsilon}{\mathfrak{L}} MM_f \sqrt[q]{\frac{2(e^{\frac{q\delta}{2}} - 1)}{q\delta}} \sum_{k=1}^{\infty} e^{-\frac{\delta(\tau_\varepsilon+k)}{2}} + \frac{\varepsilon}{3} + \frac{\varepsilon}{6} \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{6} < \varepsilon.
\end{aligned}$$

This implies that the set $\{\Gamma x : x \in \mathbb{B}\}$ is equicontinuous.

Now we denote the closed convex hull of $\Gamma\mathbb{B}$ by $\overline{conv}\Gamma\mathbb{B}$. Since $\Gamma\mathbb{B} \subset \mathbb{B}$ and $\mathbb{B}$ is closed convex, $\overline{conv}\Gamma\mathbb{B} \subset \mathbb{B}$. Thus, $\Gamma(\overline{conv}\Gamma\mathbb{B}) \subset \Gamma\mathbb{B} \subset \overline{conv}\Gamma\mathbb{B}$. This implies that Γ is a continuous mapping from $\overline{conv}\Gamma\mathbb{B}$ to $\overline{conv}\Gamma\mathbb{B}$. It is easy to verify that $\overline{conv}\Gamma\mathbb{B}$ has the properties (i) and (ii). More explicitly, $\{x(t) : x \in \overline{conv}\Gamma\mathbb{B}\}$ is relatively compact in $\mathbb{X}$ for each $t \in \mathbb{R}$, and $\overline{conv}\Gamma\mathbb{B} \subset BC(\mathbb{R}, \mathbb{X})$ is uniformly bounded and equicontinuous. By the Ascoli-Arzelà theorem, the restriction of $\overline{conv}\Gamma\mathbb{B}$ to every compact subset K' of $\mathbb{R}$, namely $\{x(t) : x \in \overline{conv}\Gamma\mathbb{B}\}_{x\in K'}$ is relatively compact in $C(K', \mathbb{X})$. Thus, by the conditions (H4)(iii) and Lemma 3.2 we deduce that $\Gamma : \overline{conv}\Gamma\mathbb{B} \to \overline{conv}\Gamma\mathbb{B}$ is a compact operator. So by the Schauder fixed point theorem, we conclude that there is a fixed point $x(\cdot)$ for Γ in $\overline{conv}\Gamma\mathbb{B}$. That is Eq. (1.1) has at least one weighted pseudo almost periodic mild solution $x \in \mathbb{B}$. This completes the proof. □

Proposition 3.4. *Assume that $\{T(t)\}_{t\geq 0}$ is an exponentially stable C_0-semigroup. Let $f \in WPAPS^p(\mathbb{X})$ and there exists a positive number L such that*

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\| \quad \textit{for all} \quad t \in \mathbb{R}, x, y \in \mathbb{X}.$$

If $\frac{ML\|B\|_{\mathcal{L}(\mathbb{X})}}{\delta} < 1$, then (1.1) *has a unique weighted pseudo almost periodic mild solution on $\mathbb{R}$.*

Proof. We can easily obtain this result by using Lemma 3.2 combined with the Banach fixed point theorem. □

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REFERENCES

[1] C.Y. Zhang, *Pseudo almost periodic solutions of some differential equations*, J. Math. Anal. Appl. **151** (1994), 62–76.

[2] C.Y. Zhang, *Integration of vector-valued pseudo almost periodic functions*, Proc. Amer. Math. Soc. **121** (1994), 167–174.

[3] C.Y. Zhang, *Pseudo almost periodic solutions of some differential equations II*, J. Math. Anal. Appl. **192** (1995), 543–561.

[4] R.P. Agarwal, T. Diagana, E. Hernández, *Weighted pseudo almost periodic solutions to some partial neutral functional differential equations*, J. Nonlinear Convex Anal. **8** (2007) 3, 397–415.

[5] T. Diagana, *Weighted pseudo almost periodic functions and applications*, C. R. Acad. Sci. Paris Ser. I **343** (2006) 10, 643–646.

[6] T. Diagana, *Weighted pseudo almost periodic solutions to some differential equations*, Nonlinear Anal. **68** (2008) 8, 2250–2260.

[7] T. Diagana, *Existence of weighted pseudo almost periodic solutions to some classes of hyperbolic evolution equations*, J. Math. Anal. Appl. **350** (2009) 18–28.

[8] G.M. N'Guérékata, A. Pankov, *Stepanov-like almost automorphic functions and monotone evolution equations*, Nonlinear Anal. TMA **68** (2008) 2658–2667.

[9] T. Diagana, G.M. Mophou, G.M. N'Guérékata, *Existence of weighted pseudo almost periodic solutions to some classes of differential equations with S^p-weighted pseudo almost periodic coefficients*, Nonlinear Anal. TMA **72** (2010) 1, 430–438.

[10] E. Hernández, H.R. Henríquez, *Existence of periodic solutions of partial neutral functional differential equations with unbounded delay*, J. Math. Anal. Appl. **221** (1998) 499–522.

[11] H.R. Henríquez, C.H. Vasquez, *Almost periodic solutions of abstract retarded functional differential equations with unbounded delay*, Acta Appl. Math. **57** (1999) 105–132.

[12] S. Abbas, D. Bahuguna, *Almost periodic solutions of neutral functional differential equations*, Comput. Math. Appl. **55** (2008) 2593–2601.

[13] H.X. Li, F.L. Huang, J.Y. Li, *Composition of pseudo almost periodic functions and semilinear differential equations*, J. Math. Anal. Appl. **255** (2001) 436–446.

[14] T. Diagana, C.M. Mahop, G.M. N'Guérékata, B. Toni, *Existence and uniqueness of pseudo almost periodic solutions to some classes of semilinear differential equations and applications*, Nonlinear Anal. **64** (2006) 2442–2453.

[15] T. Diagana, *Existence and uniqueness of pseudo almost periodic solutions to semilinear differential equations and applications*, Nonlinear Anal. **66** (2007) 228–240.

[16] H.S. Ding, J. Liang, G.M. N'Guérékata, T.J. Xiao, *Pseudo almost periodicity to some nonautonomous evolution equations with delay*, Nonlinear Anal. TMA **67** (2007) 1412–1218.

[17] H.S. Ding, J. Liang, G.M. N'Guérékata, T.J. Xiao, *Mild pseudo almost periodic solutions of nonautonomous semilinear evolution equations*, Math. Comput. Modelling **45** (2007) 579–584.

[18] E. Hernández, H.R. Henríquez, *Pseudo almost periodic solutions for non-autonomous neutral differential equations with unbounded delay*, Nonlinear Anal. RWA **9** (2008) 430–437.

[19] C. Cuevas, E. Hernández, *Pseudo almost periodic solutions for abstract partial functional defferential equations*, Appl. Math. Lett. **22** (2009) 4, 534–538.

[20] Z.R. Hu, Z. Jin, *Stepanov-like pseudo almost periodic mild solutions to perturbed non-autonomous evolution equations with infinite delay*, Nonlinear Anal. **71** (2009) 5381–5391.

[21] Z.H. Zhao, Y.K. Chang, W.S. Li, *Asymptotically almost periodic, almost periodic and pseudo almost periodic mild solutions for neutral differential equations*, Nonlinear Anal. RWA **11** (2010) 3037–3044.

[22] N. Boukli-Hacene, K. Ezzinbi, *Weighted pseudo almost periodic solutions for some partial functional differential equations*, Nonlinear Anal. TMA **71** (2009) 9, 3612–3621.

[23] L.W. Zhang, Y.T. Xu, *Weighted pseudo almost periodic solutions of a class of abstract differential equations*, Nonlinear Anal. TMA **71** (2009) 9, 3705–3714.

[24] R.P. Agarwal, B. de Andrade, C. Cuevas, *Weighted pseudo almost periodic solutions of a class of semilinear fractional differential equations*, Nonlinear Anal. RWA **11** (5) (2010), 3532–3554.

[25] H.X. Li, L.L. Li, *Stepanov-like pseudo almost periodicity and semilinear differential equations with uniform continuity*, Reaults. Math. 59 (2011), 43–61, DOI:10.1007/s00025-010-0050-4.

[26] Y.K. Chang, Z.H. Zhao, J.J. Nieto, *Pseudo almost automorphic and weighted pseudo almost automorphic mild solutions to semi-linear differential equations in Hilbert spaces*, Revista Matemática Complutense (2010), DOI: 10.1007/s13163-010-0047-2.

[27] A.M. Fink, *Almost periodic differential equations*, [in:] Lecture Notes in Mathematics, vol. 377, Springer-Verlag, New York, Berlin, 1974.

[28] G.M. N'Guérékata, *Almost automorphic and almost periodic functions in abstract space*, Kluwer Academic Plenum Publishers, New York, London, Moscow, 2001.

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