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STOCHASTIC DIFFERENTIAL EQUATIONS FOR RANDOM MATRICES PROCESSES IN THE NONLINEAR FRAMEWORK

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Abstract. In this paper, we investigate the processes of eigenvalues and eigenvectors of a symmetric matrix valued process X_t, where X_t is the solution of a general SDE driven by a G-Brownian motion matrix. Stochastic differential equations of these processes are given. This extends results obtained by P. Graczyk and J. Malecki in [*Multidimensional Yamada-Watanabe theorem and its applications to particle systems*, J. Math. Phys. 54 (2013), 021503].

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1. INTRODUCTION

Random Matrix Theory is an active research area of modern Mathematics with input from Mathematical and Theoretical Physics, Mathematical Analysis and Probability. Now we will talk about the origins of random matrix theory in mathematical statistics, common knowledge out of the 1928 paper of Wishart on correlation matrices. The real start of the field is usually attributed to highly influential papers by Eugene Wigner in the 1950's motivated by applications in Nuclear Physics [1].

Recently Graczyk and Malecki in 2013 [5] derived, in a general context, a system of SDEs for the eigenvalues and the eigenvectors for a solution X_t valued in the space of symmetric $n \times n$ matrices, of an SDE driven by a Brownian motion matrix of dimension $n \times n$. Under some conditions on the SDE satisfied by X_t, they established the existence and the uniqueness of the stochastic differential equations of eigenvalues and eigenvectors and shown that the eigenvalues never collide.

In recent decades, the theory and methodology of nonlinear expectation have been well developed and received much attention in some application fields such as finance, risk measure and control. A typical example of the nonlinear expectation, called G-expectation was introduced by Peng [7]. Under this G-expectation framework a new type of Brownian motion called G-Brownian motion was constructed and the related stochastic calculus was established.

The aim of this paper is to bring together the notion of random matrices and G-stochastic calculus to study SDEs of eigenvalues and eigenvectors for a matrix process. Namely, we consider the following general G-SDE

$$dX_t = g(X_t)dB_t h(X_t) + h(X_t)dB_t^T g(X_t) + a(X_t)dt + c(X_t)d\left\langle B\right\rangle_t ,$$

where B_t is a G-Brownian motion matrix of dimension $n \times n$, the matrix stochastic process X_t takes values in the space of symmetric $n \times n$ matrices and the function $g, h, a, c : \mathbb{R} \to \mathbb{R}$ act on the spectrum of X_t. The main difficulties lie in the fact that the G-expectation is not linear and that $\langle B\rangle$ is not a deterministic process. The notion of independence of random variables with respect to a non linear expectation being delicate, so we assume that $\left\langle B^{ij}, B^{kl}\right\rangle = \delta_{ik}\delta_{jl}b^j$ for some increasing process b^j. Like in [5], we derive a system of SDEs for the eigenvalues and the eigenvectors of the solution of X_t, which is guaranteed by Lipschitz and linear growth conditions, and prove that the eigenvalues never collide.

The rest of the paper is organized as follows. In Section 2, we recall the G-expectation framework. In Section 3 we adapt this concept according to our objective. Besides, we give the related properties of the G-Brownian motion matrix and the G-Itô's formula. In Section 4, we give our main results. In Section 5, we state the existence and uniqueness theorem of solutions of stochastic differential equations driven by G-Brownian motion matrix.

2. PRELIMINARIES

In this section, we introduce some notations and preliminaries of the theory of sublinear expectations and the related G-stochastic analysis, which will be needed in what follows. More details of this section can be found in Peng [7, 9, 10]. Let Ω be a given nonempty set and $\mathcal{H}$ a linear space of real valued functions defined on Ω such that $1 \in \mathcal{H}$ and $|X| \in \mathcal{H}$, for all $X \in \mathcal{H}$.

Definition 2.1. A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: For all $X, Y \in \mathcal{H}$, we have:

1. monotonicity: If $X \geq Y$, then $\hat{E}[X] \geq \hat{E}[Y]$;
2. preservation of constants: $\hat{E}[c] = c$, for all $c \in \mathbb{R}$;
3. subadditivity: $\hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y]$;
4. positive homogeneity: $\hat{E}\left[\lambda X\right] = \lambda \hat{E}\left[X\right]$, for all $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

Remark 2.2. $\mathcal{H}$ is considered as the space of random variables on Ω.

Let us now consider a space of random variables $\mathcal{H}$ with the additional property of stability with respect to bounded Lipschitz functions. More precisely, we suppose, if $X_i \in \mathcal{H}$, $i = 1, \dots, d$, then

$$\varphi(X_1, X_2, \dots, X_d) \in \mathcal{H}, \text{ for all } \varphi \in C_{b,Lip}(\mathbb{R}^d),$$

where $C_{b,Lip}(\mathbb{R}^d)$ denotes the space of all bounded Lipschitz functions on $\mathbb{R}^d$.

Definition 2.3. In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, a random vector $Y = (Y_1, Y_2, \dots, Y_n)$, $Y_i \in \mathcal{H}$, is said to be independent under $\hat{E}$ from another random vector $X = (X_1, X_2, \dots, X_m)$, $X_i \in \mathcal{H}$, if for each test function $\varphi \in C_{b,Lip}(\mathbb{R}^{m+n})$ we have

$$\hat{E}\left[\varphi(X,Y)\right] = \hat{E}\left[\hat{E}\left[\varphi(X,Y)\right]\Big|_{x=X}\right].$$

Definition 2.4. Let X_1 and X_2 be two $n-$dimensional random vectors defined respectively on the sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if

$$\hat{E}_1\left[\varphi(X_1)\right] = \hat{E}_2\left[\varphi(X_2)\right] \quad \text{for all } \varphi \in C_{b,Lip}(\mathbb{R}^n).$$

After the above basic definition we introduce now the central notion of G-normal distribution.

Definition 2.5. A $d-$dimensional random vector $X = (X_1, \dots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called G-normal distributed if for each $a, b \geq 0$:

$$aX + b\overline{X} \overset{d}{=} \sqrt{a^2 + b^2}X,$$

where $\overline{X}$ is an independent copy of X, and

$$G(A) := \frac{1}{2}\hat{E}\left[\langle AX, X\rangle\right] : \mathbb{S}_d \to \mathbb{R},$$

here $\mathbb{S}_d$ denotes the collection of $d \times d$ symmetric matrices.

By [7], we know that $X = (X_1, \dots, X_d)$ is G-normal distributed if and only if $u(t,x) := \hat{E}\left[\varphi(X + \sqrt{t}X)\right]$, $(t,x) \in [0,\infty) \times \mathbb{R}^d$, $\varphi \in C_{b,Lip}(\mathbb{R}^d)$, is the unique viscosity solution of the following G-heat equation:

$$\begin{cases} \partial_t u(t,x) = G(Du(t,x)), (t,x) \in [0,\infty) \times \mathbb{R}^d, \\ u(0,x) = \varphi(X), \end{cases}$$

where $Du(t,x)$ is the Hessian of $u(t,x)$.

The function $G(\cdot) : \mathbb{S}_d \to \mathbb{R}$ is a monotonic sublinear functional on $\mathbb{S}_d$, from which we can deduce that there exists a bounded, convex and closed subset $\Sigma \in \mathbb{S}_d^+$ the collection such that

$$G(A) = \frac{1}{2}\sup_{B \in \Sigma} tr(AB).$$

In this context, the set $\Gamma = \left\{Q \in \mathbb{R}^{d \times d} : QQ^T \in \Sigma\right\}$ captures the uncertainty of the probability distribution (variance uncertainty) of the G-distributed random vector X. Note that if $d = 1$, X has no mean uncertainty. We write $X \sim N(0; \Sigma)$.

Remark 2.6. When $d = 1$, Σ is an interval that is $\Sigma = \left[\underline{\sigma}^2; \overline{\sigma}^2\right]$ with $0 \leq \underline{\sigma} \leq \overline{\sigma}$. Here $G = G_{\overline{\sigma},\underline{\sigma}}$ is the following sublinear function parameterized by $\underline{\sigma}$ and $\overline{\sigma}$:

$$G(\alpha) = \frac{1}{2}(\overline{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-), \quad \alpha \in \mathbb{R},$$

Recall that $\alpha^+ = \max\{0, \alpha\}$ and $\alpha^- = -\min\{0, \alpha\}$. In fact $\overline{\sigma}^2 = \hat{E}\left[X^2\right]$ and $\underline{\sigma}^2 = -\hat{E}\left[-X^2\right]$ (see [7, 11]).

Definition 2.7. A $d-$dimensional process $B = (B_t)_{t\geq 0} \subset \mathcal{H}$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called a G-Brownian motion if the following properties are satisfied:

a) $B_0 = 0$;
b) for each $t, s \geq 0$, the increment $B_{t+s} - B_t$ is $N(0; s\Sigma)-$distributed and is independent from $(B_{t_1}, \ldots, B_{t_n})$, for all $n \in \mathbb{N}$ and $0 \leq t_1 \leq \ldots \leq t_n \leq t$.

Note that $\langle a, B_t\rangle$ is a real $G_{\underline{\sigma_a},\overline{\sigma_a}}-$Brownian motion for each $a \in \mathbb{R}^d$, where $\langle\cdot,\cdot\rangle$ is the Euclidian inner product of $\mathbb{R}^d$, $\overline{\sigma_a}^2 = \hat{E}(\langle a, B_1\rangle^2)$ and $\underline{\sigma_a}^2 = -\hat{E}(-\langle a, B_1\rangle^2)$ (for more details, see [10]).

3. G-MATRICIAL STOCHASTIC CALCULUS

In the following we will identify each $n \times n$ matrix to a vector of n^2 dimension. Let us consider $\Omega = C_0(\mathbb{R}^{n\times n})$ the set of all $\mathbb{R}^{n\times n}-$valued continuous functions $(\omega_t)_{t\in\mathbb{R}^+}$ with $\omega_0 = 0$, where $\mathbb{R}^{n\times n}$ is the space of $n \times n$ matrix, equipped with the distance

$$\rho(\omega^1, \omega^2) = \sum_{i=1}^{\infty} 2^{-i}\left[(\max_{t\in[0,i]}\left|\omega_t^1 - \omega_t^2\right|) \wedge 1\right], \quad \omega^1, \omega^2 \in \Omega.$$

We denote by $B(\Omega)$ the Borel $\sigma-$algebra on Ω. We also set, for each $t \in [0, \infty)$, $\Omega_t := \{\omega_{.\wedge t} : \omega \in \Omega\}$. The spaces of Lipschitzian functions on Ω are denoted by:

$$\begin{aligned} Lip(\Omega_t) &= \left\{\varphi(B_{t_1\wedge t}, \ldots, B_{t_d\wedge t}) : t_1, \ldots, t_d \in [0; \infty),\ \varphi \in C_{b,Lip}(\mathbb{R}^{n\times n})^d\right\}, \\ Lip(\Omega) &= \bigcup_{n=1}^{\infty} Lip(\Omega_n). \end{aligned}$$

Here we use the space of all Lipschitzian and bounded functions $C_{b,Lip}(\mathbb{R}^{n\times n})^d$ in our framework only for convenience. In general $Lip(\Omega_t)$, $Lip(\Omega)$ can be replaced by the following spaces of functions defined on $\mathbb{R}^{n\times n}$:

– $L^0(\Omega)$: the space of all $B(\Omega)$-measurable real valued functions on Ω;
– $L^0(\Omega_t)$: the space of all $B(\Omega_t)$-measurable real valued functions on Ω;
– $L_b(\Omega)$: the space of all bounded elements in $L^0(\Omega)$;
– $L_b(\Omega_t)$: the space of all bounded elements in $L^0(\Omega_t)$.

Let $T > 0$ be a fixed time. We denote by $L_G^p(\Omega_T)$, $p \geq 1$, the completion of G-expectation space $Lip(\Omega_T)$ with respect to the norm $\|X\|_p := \hat{E}\left[|X|^p\right]^{1/p}$, $1 \leq p < \infty$. Let $L_G^p(\Omega)$ be the Banach space defined as the closure of

$$\mathcal{H} := \left\{\varphi(\omega_{t_1}, \ldots, \omega_{t_d}) : \varphi \in C_{b,Lip}(\mathbb{R}^{n\times n})^d, 0 \leq t_1 < \ldots < t_d, d \geq 1\right\}.$$

As in [9–11], we can construct a nonlinear expectation $\hat{E}$ on $\mathcal{H}$ under which the coordinate process (i.e. $B_t(\omega) = \omega_t$) is a G-Brownian motion matrix and the conditional expectation $\hat{E}(\cdot \mid \Omega_t)$, which is continuous on $L_G^p(\Omega_T)$. Thus (B_t^{ij}) is a $G_{\underline{\sigma_{ij}},\overline{\sigma_{ij}}}$–Brownian motion where $\overline{\sigma_{ij}}^2 = \hat{E}\left[(B_1^{ij})^2\right]$ and $\underline{\sigma_{ij}}^2 = -\hat{E}\left[-(B_1^{ij})^2\right]$ for each $i, j \in \overline{1, n}$.

Let us point out that the space $C_b(\Omega)$ of the bounded continuous functions on Ω is a subset of $L_G^1(\Omega)$. Moreover, there exists a weakly compact family P of probability measures on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\hat{E}[\cdot] = \sup_{P\in\mathcal{P}} E^P[\cdot],$$

where E^P stands for the expectation with respect to the probability P (see [11]). We introduce the natural capacity $c(\cdot)$ associated to P defined by

$$c(A) := \sup_{P\in\mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

Definition 3.1. A set $A \subseteq \Omega$ is polar if $c(A) = 0$. A property holds "quasi-surely" (q.s., for short) if it holds outside a polar set.

Definition 3.2. A process $(M_t)_{0\leq t\leq T}$ is called G-martingale if for each t, $M_t \in L_G^1(\Omega_t)$ and for each $s \in [0, t]$ we have $\hat{E}(M_t \mid \Omega_s) = M_s$, where $\hat{E}(\cdot \mid \Omega_t)$ is a continuous mapping on $Lip(\Omega_T)$ endowed with the norm $\|\cdot\|_{1,G}$. Therefore, it can be extended continuously to $L_G^p(\Omega_T)$.

For each $p \geq 1$, consider the following space $M_G^{0,p}(0, T)$ of simple type of processes, that is

$$\eta := \eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)\mathbf{1}_{[t_j,t_{j+1})}(t) \text{ for } 0 = t_0 < \ldots < t_N = T,$$

where $\xi_j \in L_G^p(\Omega_{t_j})$, $j = 0, \ldots, N-1$. Denote by $M_G^p(0, T)$ the completion of $M_G^{0,p}(0, T)$ under the norm

$$\|\eta\|_{M_G^p(0,T)} = \left|\int_0^T \hat{E}\left[|\eta(t)|^p\right] dt\right|^{1/p}.$$

For two processes $\eta \in M_G^2(0, T)$ and $\xi \in M_G^1(0, T)$, the G-Itô integrals $(\int_0^t \eta_s dB_s^{ij})_{0\leq t\leq T}$, which is a G-martingale and $(\int_0^t \xi_s d\left\langle B^{ij}, B^{kl}\right\rangle_s)_{0\leq t\leq T}$ are well defined (see [2, 7, 10, 12]), where the quadratic co-variation process $\left\langle B^{ij}, B^{kl}\right\rangle$ is the non deterministic process formulated in $L^2_G(\Omega_t)$ defined by

$$\left\langle B^{ij}, B^{kl}\right\rangle_t := B^{ij}_t B^{kl}_t - \int_0^t B^{ij}_s dB^{kl}_s - \int_0^t B^{kl}_s dB^{ij}_s.$$

We write $\left\langle B^{ij}\right\rangle$ instead of $\left\langle B^{ij}, B^{ij}\right\rangle$ the quadratic variation of B^{ij}. In fact, $\left\langle B^{ij}, B^{kl}\right\rangle_t$ can be regarded as the limit in $L^2_G(\Omega_t)$ of

$$\sum_{p=1}^{N}(B^{ij}_{t^m_{p+1}} - B^{ij}_{t^m_p})(B^{kl}_{t^m_{p+1}} - B^{kl}_{t^m_p}),$$

where $\{0 = t^m_0 < t^m_1 < \ldots < t^m_N = T\}$ is a sequence of partitions of $[0, T]$ such that $\max_p \left|t^m_{p+1} - t^m_p\right|$ tends to 0 when m goes to infinity. It was shown in [12] that $\underline{\sigma_{ij}}^2 t \leq \left\langle B^{ij}\right\rangle_t \leq \overline{\sigma_{ij}}^2 t$.

For the following generalized Itô formula (see [8] for the vectorial case), we use Einstein's notation.

Theorem 3.3. *Let $\varphi \in C^2(\mathbb{R}^{n\times n})$ and its first and second derivatives are in $C_{b,Lip}(\mathbb{R}^{n\times n})$. Let $X = (X^{ij})$ be a matrix process on $[0, T]$ with the form*

$$X^{pq}_t = X^{pq}_0 + \int_0^t \alpha^{pq}(s)ds + \int_0^t \theta^{pq}_{ijkl}(s)d\left\langle B^{ij}, B^{kl}\right\rangle_s + \int_0^t \beta^{pq}_{kl}(s)dB^{kl}_s,$$

where $\alpha^{pq}, \theta^{pq}_{ijkl} \in M^1_G(0, T)$ and $\beta^{pq}_{kl} \in M^2_G(0, T)$. Then for each $t \in [0, T]$, we have, q.s.,

$$\begin{aligned}
&\varphi(X_t) - \varphi(X_0) \\
&= \int_0^t \partial_{x^{pq}}\varphi(X_u)\beta^{pq}_{kl}(u)dB^{kl}_u + \int_0^t \partial_{x^{pq}}\varphi(X_u)\alpha^{pq}(u)du \\
&\quad + \int_0^t \left[\partial_{x^{pq}}\varphi(X_u)\theta^{pq}_{ijkl}(u) + \frac{1}{2}\partial^2_{x^{p'q'}x^{pq}}\varphi(X_u)\beta^{pq}_{ij}(u)\beta^{p'q'}_{kl}(u)\right] d\left\langle B^{ij}, B^{kl}\right\rangle_u .
\end{aligned}$$

Note that this formula remains valid if X is not a square matrix.

In the following we use the notation

$$dX^{pq}_t dX^{mn}_t = \sum_{i,j,k,l} \beta^{pqij}_t \beta^{mnkl}_t d\left\langle B^{ij}, B^{kl}\right\rangle_t .$$

We have then $d\left\langle B^{ij}, B^{kl}\right\rangle = dB^{ij}dB^{kl}$.

Let $\mathbb{S}_n$ be the collection of symmetric $n-$dimensional matrices identified with $\mathbb{R}^{n(n+1)/2}$. Recall that if $g : \mathbb{R} \to \mathbb{R}$, $X \in \mathbb{S}_n$ and $X = H\Lambda H^T$ be the factorization with H an orthonormal matrix and Λ a diagonal one, then $g(X) = Hg(\Lambda)H^T$. Let X_t be a stochastic process with values in $\mathbb{S}_n$ such that $X_0 \in \widetilde{\mathbb{S}}_n$, the set of symmetric matrices with n different eigenvalues. Let $\Lambda_t = diag(\lambda_i(t))$ be the diagonal matrix of eigenvalues of X_t ordered increasingly: $\lambda_1(t) \leq \lambda_2(t) \leq \ldots \leq \lambda_n(t)$ and H_t an orthonormal matrix of eigenvectors of X_t. Matrices Λ and H may be chosen as smooth functions of X until the first collision time $\tau = \inf\{t : \lambda_i(t) = \lambda_j(t) \text{ for some } i \neq j\}$.

As in the classical case, we define the Stratonovich differential $\circ$ for two matrices X and Y:

$$X \circ dY = XdY + \frac{1}{2}dXdY \text{ and } dX \circ Y = dXY + \frac{1}{2}dXdY.$$

Proposition 3.4. *We have for each matrices process X, Y defined as in Theorem 3.3:*

(i) *the integration formula by parts holds:*

$$d(XY) = XdY + dXY + dXdY,$$

where $dXdY$ is the classical matricial product,

(ii)

$$\begin{aligned} d(XY) &= dX \circ Y + X \circ dY, \\ dX \circ (YZ) &= (dX \circ Y) \circ Z, \\ (X \circ dY)^T &= dY^T \circ X^T. \end{aligned}$$

Proof. By using the theorem 3.3 with $\varphi(x, y) = xy$ we obtain that

$$d(X_t^{pq} Y_t^{mn}) = dX_t^{pq} Y_t^{mn} + X_t^{pq} dY_t^{mn} + dX_t^{pq} dY_t^{mn},$$

which implies (i).

(ii) follows from (i) and the definition of the Stratonovich differential. □

4. MAIN RESULTS

In the rest of this paper, we assume that B satisfies the following assumption:

(A) There exist an increasing real process b^j such that $\left\langle B^{ij}, B^{kl}\right\rangle_t = \delta_{ik}\delta_{jl} b_t^j$ $q.s.$ for each $i, j, k, l \in \overline{1, n}$, where δ_{uv} is the Kronecker symbol.

We have then $\underline{\sigma}^2 t \leq b_t^j \leq \overline{\sigma}^2 t$, where $\overline{\sigma} := \max\limits_{i,j} \overline{\sigma_{ij}}$ and $\underline{\sigma} := \min\limits_{i,j} \underline{\sigma_{ij}}$. Note that in the classical case the assumption (A) is satisfied with $b_t^j = t$.

Let us consider the general G-stochastic differential equation defined by

$$dX_t = g(X_t)dB_t h(X_t) + h(X_t)dB_t^T g(X_t) + a(X_t)dt + c(X_t)d\left\langle B\right\rangle_t \tag{4.1}$$

with $g, h, a, c : \mathbb{R} \to \mathbb{R}$, and $X_0 \in \widetilde{\mathbb{S}}_n$, where the quadratic variation $d\langle B\rangle$ of the matrix B is defined by $d\langle B\rangle := dBdB$. Thus, according to the assumption (A), $d\langle B\rangle$ is diagonal matrix such that $d\langle B\rangle^{ij} = \delta_{ij} db^i$.

Now we are able to state our main result. Note that in our model, the stochastic differential equation studied (4.1) behaves as in the linear case. The techniques used are inspired by the linear case, where the G-Brownian motion plays the role of a classical Brownian motion.

Theorem 4.1. *Let X_t be a solution of the equation* (4.1) *such that $X_0 \in \widetilde{\mathbb{S}}_n$. Then there exists a G-real Brownian motion W^i (resp. β^{ij}) such that $\langle W^i, W^j\rangle = \delta_{ij} b^j$ (resp. $\langle \beta^{ij}, \beta^{kl}\rangle = \delta_{ik}\delta_{jl} b^j$) for each $i, j, k, l \in \overline{1, n}$ such that for $t < \tau$ the eigenvalues process Λ_t and the eigenvectors process H_t are solutions of the following system:*

$$d\lambda_i = 2g(\lambda_i)h(\lambda_i)\sum_k H^{ki} dW^k + a(\lambda_i)dt + dV^{ii} \tag{4.2}$$

$$\begin{aligned} dH^{ij} = &\sum_{k\neq j} \frac{H^{ik}}{\lambda_j - \lambda_k}\Big\{ \big[g(\lambda_k)h(\lambda_j)(d\beta H)^{jj} + g(\lambda_j)h(\lambda_k)(d\beta H)^{kk}\big] + dV^{kj}\Big\} \\ &- \frac{1}{2}\sum_k H^{ik} dQ^{kj}, \end{aligned} \tag{4.3}$$

where

$$dV^{ij} = \delta_{ij} c(\lambda_i)\sum_k (H^{ki})^2 db^k + dR^{ij}$$

with

$$\begin{aligned} dR^{ij} = \sum_{k\neq j} \tfrac{1}{\lambda_j - \lambda_k}\Big[&(\delta_{ik} g^2(\lambda_k)h(\lambda_k)h(\lambda_j) + g^2(\lambda_k)h(\lambda_i)h(\lambda_j))\sum_l H^{li}H^{lj} db^l \\ &+ \delta_{ij} g^2(\lambda_j)h^2(\lambda_k)\sum_l (H^{lk})^2 db^l\Big] \end{aligned}$$

and

$$\begin{aligned} dQ^{kj} = \sum_{l\neq k, l\neq j} \frac{1}{(\lambda_l - \lambda_k)(\lambda_l - \lambda_j)}\Big[&\delta_{kj} g^2(\lambda_k)h^2(\lambda_l)\sum_p (H^{pl})^2 db^p \\ &+ g^2(\lambda_l)h(\lambda_k)h(\lambda_j)\sum_p H^{pk}H^{pj} db^p\Big]. \end{aligned}$$

Proof. Firstly, to simplify the notation we write Ψ^{ij} instead of $\Psi(X_t)^{ij}$ for $\Psi = g, g^2, h, a$ and c. Let A be the skew-symmetric matrix defined by $dA = H^T \circ dH$ and let the matrix $dN := H^T \circ dX \circ H$. By applying the G-integration formula by parts to $\Lambda = H^T X H$, we get $d\Lambda = dN - dA \circ \Lambda + \Lambda \circ dA$. Now observe that the process $\Lambda \circ dA - dA \circ \Lambda$ is zero on the diagonal. Consequently $d\lambda_i = dN^{ii}$ and $0 = dN^{ij} + (\lambda_i - \lambda_j)dA^{ij}$, when $i \neq j$ and so

$$dA^{ij} = \frac{1}{\lambda_j - \lambda_i} dN^{ij} \text{ for } i \neq j. \tag{4.4}$$

We have

$$dX_t^{ij} = \sum_{p,q} g^{ip} dB_t^{pq} h^{qj} + \sum_{p,q} h^{ip} dB_t^{qp} g^{qj} + a^{ij} dt + \sum_{p,q} c^{ip} d \left\langle B^{pq}, B^{qj} \right\rangle_t ,$$

and then

$$\begin{aligned} dX_t^{ij} dX_t^{km} &= \sum_{p,q} [g^{ip} h^{qj} g^{kp} h^{qm} + g^{ip} h^{qj} g^{pm} h^{kq}] db_t^q \\ &\quad + \sum_{p,q} [h^{ip} g^{qj} g^{kq} h^{pm} + h^{ip} g^{qj} h^{kp} g^{qm}] db_t^p \\ &= \sum_l \left[(g^2)^{ik} h^{lj} h^{lm} + (g^2)^{im} h^{lj} h^{kl} + (g^2)^{kj} h^{il} h^{lm} + (g^2)^{jm} h^{il} h^{kl} \right] db_t^l. \end{aligned}$$

Finally we get

$$\begin{aligned} dX_t^{ij} dX_t^{km} = \sum_l \big[&(g^2)^{ik} h^{lj} h^{lm} + (g^2)^{kj} h^{li} h^{lm} \\ &+ (g^2)^{im} h^{lj} h^{lk} + (g^2)^{jm} h^{li} h^{lk} \big] db_t^l. \end{aligned} \tag{4.5}$$

A simple calculation of dN gives

$$dN = H^T dX H + \frac{1}{2} H^T dX dH + \frac{1}{2} dH^T dX H \tag{4.6}$$

and consequently the G-martingale part of dN equals the G-martingale part of $H^T dX H$. We have

$$dN^{ij} dN^{km} = (H^T dX H)^{ij} (H^T dX H)^{km},$$

which equals

$$\sum_{p,q,p',q'} H^{pi} dX^{pq} H^{qj} H^{p'k} dX^{p'q'} H^{q'm},$$

and taking into account the formula (4.5) with (p, q, p', q') instead of (i, j, k, m), we get

$$\begin{aligned}
dN^{ij}dN^{km} &= \sum_{l}\sum_{p,q,p',q'} H^{pi}H^{qj}H^{p'k}H^{q'm}\Big[\left(g^2\right)^{pp'}h^{lq}h^{lq'} + (g^2)^{p'q}h^{lp}h^{lq'} \\
&\quad +(g^2)^{pq'}h^{lq}h^{lp'} + (g^2)^{qq'}h^{lp}h^{lp'}\Big]\, db^l \\
&= \underbrace{\sum_{l}\left[\sum_{p,p'} H^{pi}(g^2)^{pp'}H^{p'k}\sum_{q,q'} H^{qj}h^{lq}h^{lq'}H^{q'm}\right] db^l}_{(I)} \\
&\quad + \underbrace{\sum_{l}\left[\sum_{q,p'} H^{qj}(g^2)^{p'q}H^{p'k}\sum_{q',p} H^{pi}h^{lp}h^{lq'}H^{q'm}\right] db^l}_{(II)} \\
&\quad + \underbrace{\sum_{l}\left[\sum_{p,q'} H^{pi}(g^2)^{pq'}H^{q'm}\sum_{q,p'} H^{qj}h^{lq}h^{lp'}H^{p'k}\right] db^l}_{(III)} \\
&\quad + \underbrace{\sum_{l}\left[\sum_{q,q'} H^{qj}(g^2)^{qq'}H^{q'm}\sum_{p,p'} H^{pi}h^{lp}h^{lp'}H^{p'k}\right] db^l}_{(IV)}.
\end{aligned}$$

Now observe that

$$\sum_{p,p'} H^{pi}(g^2)^{pp'}H^{p'k} = (H^T g^2 H)^{ik} = g^2(\Lambda)^{ik} = \delta_{ik}g^2(\lambda_k),$$

which implies that

$$\begin{aligned}
(I) &= \delta_{ik}g^2(\lambda_k)\sum_{q,q'} H^{qj}(\sum_{l} h^{lq}h^{lq'}db^l)H^{q'm} \\
&= \delta_{ik}g^2(\lambda_k)\sum_{q,q'} H^{qj}(h(X)d\left\langle B\right\rangle h(X))^{qq'}H^{q'm} \\
&= \delta_{ik}g^2(\lambda_k)(H^T h(X)d\left\langle B\right\rangle h(X)H)^{jm}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(II) &= \delta_{jk}g^2(\lambda_k)(H^T h(X)d\left\langle B\right\rangle h(X)H)^{im}, \\
(III) &= \delta_{im}g^2(\lambda_m)(H^T h(X)d\left\langle B\right\rangle h(X)H)^{jk}, \\
(IV) &= \delta_{jm}g^2(\lambda_m)(H^T h(X)d\left\langle B\right\rangle h(X)H)^{ik},
\end{aligned}$$

and so

$$
\begin{aligned}
dN^{ij}dN^{km} &= \delta_{ik}g^2(\lambda_k)(H^T h(X)d\langle B\rangle h(X)H)^{jm} \\
&+ \delta_{jk}g^2(\lambda_k)(H^T h(X)d\langle B\rangle h(X)H)^{im} \\
&+ \delta_{im}g^2(\lambda_m)(H^T h(X)d\langle B\rangle h(X)H)^{kj} \\
&+ \delta_{jm}g^2(\lambda_m)(H^T h(X)d\langle B\rangle h(X)H)^{ik},
\end{aligned}
$$

which implies, by using the fact that $h(X)H = Hh(\Lambda)$, that

$$
\begin{aligned}
&dN^{ij}dN^{km} \\
&= \delta_{ik}g^2(\lambda_k)(h(\Lambda)H^T d\langle B\rangle Hh(\Lambda))^{jm} + \delta_{jk}g^2(\lambda_k)(h(\Lambda)H^T d\langle B\rangle Hh(\Lambda))^{im} \\
&\quad + \delta_{im}g^2(\lambda_m)(h(\Lambda)H^T d\langle B\rangle Hh(\Lambda))^{kj} + \delta_{jm}g^2(\lambda_m)(h(\Lambda)H^T d\langle B\rangle Hh(\Lambda))^{ik}.
\end{aligned}
\tag{4.7}
$$

Since $(H^T d\langle B\rangle H)^{ij} = \sum_l H^{li}H^{lj}db^l$, then

$$
(h(\Lambda)H^T d\langle B\rangle Hh(\Lambda))^{ij} = h(\lambda_i)h(\lambda_j)\sum_l H^{li}H^{lj}db^l.
$$

It follows from (4.7) that

$$
\begin{aligned}
dN^{ij}dN^{km} &= \delta_{ik}g^2(\lambda_k)h(\lambda_j)h(\lambda_m)\sum_l H^{lj}H^{lm}db^l \\
&+ \delta_{jk}g^2(\lambda_k)h(\lambda_i)h(\lambda_m)\sum_l H^{li}H^{lm}db^l \\
&+ \delta_{im}g^2(\lambda_m)h(\lambda_k)h(\lambda_j)\sum_l H^{lk}H^{lj}db^l \\
&+ \delta_{jm}g^2(\lambda_m)h(\lambda_i)h(\lambda_k)\sum_l H^{li}H^{lk}db^l,
\end{aligned}
\tag{4.8}
$$

and so,

$$
dN^{ii}dN^{jj} = 4\delta_{ij}g^2(\lambda_i)h^2(\lambda_i)\sum_l (H^{li})^2 db^l. \tag{4.9}
$$

It follows that there exists a G-real Brownian motion W^i satisfying $\langle W^i, W^j\rangle = \delta_{ij}b^j$ such that the G-martingale part of $d\lambda_i$ equals

$$
2g(\lambda_i)h(\lambda_i)\sum_k H^{ki}dW^k \tag{4.10}
$$

Now observe that the finite variation part dF of dN is

$$
dF_t = H^T aH dt = a(\Lambda_t)dt,
$$

so that F is diagonal and

$$
dF_t^{ii} = a(\lambda_i(t))dt. \tag{4.11}
$$

Thanks to the formula (4.6) the integral part dV, with respect to db^i, of dN equals

$$\begin{aligned} dV &= H^T cd\langle B\rangle H + \frac{1}{2}(dH^T dXH + H^T dXdH) \\ &= c(\Lambda)H^T d\langle B\rangle H + \frac{1}{2}(dNdA + (dNdA)^T). \end{aligned}$$

Note that $(c(\Lambda)H^T d\langle B\rangle H)^{ij} = \delta_{ij} c(\lambda_i)\sum_k (H^{ki})^2 db^k$. We have then, if $i \neq j$,

$$\begin{aligned} (dNdA)^{ij} &= \sum_k dN^{ik} dA^{kj} = \sum_{k\neq j} \frac{1}{\lambda_j - \lambda_k} dN^{ik} dN^{kj} \\ &= \sum_{k\neq j} \frac{1}{\lambda_j - \lambda_k}\Big[\delta_{ik} g^2(\lambda_k)h(\lambda_k)h(\lambda_j)\sum_l H^{lk}H^{lj}db^l \\ &\qquad + g^2(\lambda_k)h(\lambda_i)h(\lambda_j)\sum_l H^{li}H^{lj}db^l\Big] \end{aligned}$$

and

$$(dNdA)^{ii} = \sum_{k\neq i} \frac{1}{\lambda_i - \lambda_k}\left[g^2(\lambda_k)h^2(\lambda_i)\sum_l (H^{li})^2 db^l + g^2(\lambda_i)h^2(\lambda_k)\sum_l (H^{lk})^2 db^l\right],$$

which imply that, for $i \neq j$

$$dV^{ij} = \sum_{k\neq j} \tfrac{1}{\lambda_j - \lambda_k}\left[(\delta_{ik} g^2(\lambda_k)h(\lambda_k)h(\lambda_j) + g^2(\lambda_k)h(\lambda_i)h(\lambda_j))\sum_l H^{li}H^{lj}db^l\right], \tag{4.12}$$

and

$$\begin{aligned} dV^{ii} = c(\lambda_i)\sum_k (H^{ki})^2 db^k + \sum_{k\neq i} \tfrac{1}{\lambda_i - \lambda_k}\Big[& g^2(\lambda_k)h^2(\lambda_i)\sum_l (H^{li})^2 db^l \\ & + g^2(\lambda_i)h^2(\lambda_k)\sum_l (H^{lk})^2 db^l\Big]. \end{aligned} \tag{4.13}$$

The formula (4.2) follows from (4.10), (4.11) and (4.13). In order to find a stochastic differential equation of H_t, we deduce from the definition of dA that

$$dH = H \circ dA = HdA + \frac{1}{2}dHdA = HdA + \frac{1}{2}HdAdA. \tag{4.14}$$

Thanks to the formula (4.4) we have

$$\begin{aligned} (dAdA)^{ij} &= \sum_k dA^{ik}dA^{kj} \\ &= \sum_{k\neq i, k\neq j} \frac{1}{(\lambda_k - \lambda_i)(\lambda_j - \lambda_k)}\Big[\delta_{ij} g^2(\lambda_i)h^2(\lambda_k)\sum_l (H^{lk})^2 db^l \\ &\qquad + g^2(\lambda_k)h(\lambda_i)h(\lambda_j)\sum_l H^{li}H^{lj}db^l\Big]. \end{aligned} \tag{4.15}$$

We deduce from the formula (4.8) that if $i \neq j$,

$$dN^{ij} dN^{ij} = \left[g^2(\lambda_i) h^2(\lambda_j) \sum_l (H^{lj})^2 db^l + g^2(\lambda_j) h^2(\lambda_i) \sum_l (H^{li})^2 db^l \right],$$

then the G-martingale part of dN^{ij} is

$$\begin{aligned} &\left[g(\lambda_i) h(\lambda_j) \sum_l H^{lj} d\beta^{jl} + g(\lambda_j) h(\lambda_i) \sum_l H^{li} d\beta^{il} \right] \\ &= \left[g(\lambda_i) h(\lambda_j) (d\beta H)^{jj} + g(\lambda_j) h(\lambda_i) (d\beta H)^{ii} \right], \end{aligned}$$

where $\beta := (\beta^{ij})$ is a G-Brownian motion matrix satisfying the assumption (A), so that if $i \neq j$

$$\begin{aligned} dA^{ij} &= \frac{1}{\lambda_j - \lambda_i} dN^{ij} \\ &= \frac{1}{\lambda_j - \lambda_i} \left[g(\lambda_i) h(\lambda_j) (d\beta H)^{jj} + g(\lambda_j) h(\lambda_i) (d\beta H)^{ii} \right] + dV^{ij}. \end{aligned} \tag{4.16}$$

The formula (4.3) follows from (4.14), (4.15) and (4.16). The proof is complete. □

Proposition 4.2. *Let $\Lambda = (\lambda_i)_{i=1,\dots,n}$ be a process starting from $\lambda_1(0) < \ldots < \lambda_n(0)$ and satisfying (4.2) with functions $a, c, g, h : \mathbb{R} \to \mathbb{R}$ satisfying the following hypothesis:*

(i) *There exists $C > 0$ such that*

$$|I(x) - I(y)| + \left| \overline{\sigma}^2 J^2(x) - \underline{\sigma}^2 J^2(y) \right| \leq C |x - y|, \quad \forall x, y \in \mathbb{R},$$

for $I = a, c, h^2, g^2$ and $J = g$ and h.

(ii) *There exists $K > 0$ such that*

$$\left| \overline{\sigma}^2 g^2(x) h^2(x) - \underline{\sigma}^2 g^2(y) h^2(y) \right| \leq K |x - y|^2, \quad \forall x, y \in \mathbb{R},$$

(iii) *h^2 is increasing and g^2 is decreasing on $\mathbb{R}$.*

Then we have $\tau = +\infty$ q.s., that is the distinct eigenvalues of X will never collide.

Proof. As in the proof given by [5, 6], we set

$$U = -\sum_{i<j} \log(\lambda_j - \lambda_i).$$

By using G-Itô's formula, we have

$$dU = -\sum_{i<j} \left[\frac{-d\lambda_i}{\lambda_j - \lambda_i} + \frac{d\lambda_j}{\lambda_j - \lambda_i} + \frac{1}{2} \left[\frac{-d\langle \lambda_i \rangle}{(\lambda_j - \lambda_i)^2} + \frac{-d\langle \lambda_j \rangle}{(\lambda_j - \lambda_i)^2} + 2 \frac{d\langle \lambda_i, \lambda_j \rangle}{(\lambda_j - \lambda_i)^2} \right] \right].$$

In fact

$$d\langle \lambda_i, \lambda_j \rangle = d\lambda_i d\lambda_j = 4\delta_{ij} g^2(\lambda_i) h^2(\lambda_i) \sum_k (H^{ki})^2 db^k,$$

and so

$$dU = \sum_{i<j} \frac{d\lambda_i - d\lambda_j}{\lambda_j - \lambda_i} + \frac{1}{2} \sum_{i<j} \frac{d\langle\lambda_i\rangle + d\langle\lambda_j\rangle}{(\lambda_j - \lambda_i)^2}.$$

By using the G-SDE of the eigenvalues (4.2),

$$\begin{aligned} dU &= 2\sum_{i<j} \frac{g(\lambda_i)h(\lambda_i)\sum_k H^{ki} dW^k - g(\lambda_j)h(\lambda_j)\sum_k H^{kj} dW^k}{\lambda_j - \lambda_i} \\ &\quad + \sum_{i<j} \frac{a(\lambda_i) - a(\lambda_j)}{\lambda_j - \lambda_i} dt \\ &\quad + \sum_{i<j} \frac{dV^{ii} - dV^{jj}}{\lambda_j - \lambda_i} \\ &\quad + 2\sum_{i<j} \left[\frac{g^2(\lambda_i)h^2(\lambda_i)\sum_k (H^{ki})^2 db^k + g^2(\lambda_j)h^2(\lambda_j)\sum_k (H^{kj})^2 db^k}{(\lambda_j - \lambda_i)^2} \right] \\ &= dM + dP, \end{aligned}$$

where

$$dM = 2\sum_{i<j} \frac{g(\lambda_i)h(\lambda_i)\sum_k H^{ki} dW^k - g(\lambda_j)h(\lambda_j)\sum_k H^{kj} dW^k}{\lambda_j - \lambda_i}.$$

We will show that dP is bounded on any interval $[0, T]$. To this end, we set

$$dP = dA^1 + dA^2 + dA^3$$

and

$$d^{ku} = g^2(\lambda_k)h^2(\lambda_u)\sum_l (H^{lu})^2 db^l + g^2(\lambda_u)h^2(\lambda_k)\sum_l (H^{lk})^2 db^k \quad \text{for } u = i, j,$$

where

$$A^1 = \sum_{i<j} \int_0^t \frac{a(\lambda_i) - a(\lambda_j)}{\lambda_j - \lambda_i} ds,$$

$$\begin{aligned} A_t^2 &= \sum_{i<j} \left(\int_0^t \frac{dV_s^{ii} - dV_s^{jj}}{\lambda_j - \lambda_i} + 2\int_0^t \frac{d_s^{ij}}{(\lambda_j - \lambda_i)^2} \right) \\ &= \sum_{i<j} \Bigg[\int_0^t \frac{\sum_l (c(\lambda_i)(H^{li})^2 - c(\lambda_j)(H^{lj})^2)}{\lambda_j - \lambda_i} db_s^l \\ &\qquad + \int_0^t \frac{1}{(\lambda_j - \lambda_i)} \sum_{k \neq i} \frac{d_s^{ki}}{\lambda_i - \lambda_k} - \int_0^t \frac{1}{(\lambda_j - \lambda_i)} \sum_{k \neq j} \frac{d_s^{kj}}{\lambda_j - \lambda_k} \\ &\qquad + 2\int_0^t \frac{d_s^{ij}}{(\lambda_j - \lambda_i)^2} \Bigg] \end{aligned}$$

and

$$A_t^3 = 2\sum_{i<j} \int_0^t \frac{(h^2(\lambda_j) - h^2(\lambda_i))(g^2(\lambda_j)\sum_l (H^{lj})^2 db^l - g^2(\lambda_i)\sum_l (H^{li})^2 db^l)}{(\lambda_j - \lambda_i)^2}.$$

By using the hypothesis (i) and the fact that $\sum_l (H^{li})^2 = 1$, we get

$$\begin{aligned} \left|A_t^1\right| &\leq \sum_{i<j} \int_0^t \frac{|a(\lambda_i) - a(\lambda_j)|}{|\lambda_j - \lambda_i|} ds \leq C\frac{p(p-1)}{2}T, \\ A_t^3 &\leq 2\sum_{i<j} \int_0^t \frac{(h^2(\lambda_j) - h^2(\lambda_i))(\overline{\sigma}^2 g^2(\lambda_j) - \underline{\sigma}^2 g^2(\lambda_i))}{(\lambda_j - \lambda_i)^2} ds \end{aligned}$$

and so

$$\left|A^3\right| \leq C^2 p(p-1)T.$$

We set

$$A_t^4 = \sum_{i<j} \int_0^t \frac{\sum_l (c(\lambda_i)(H^{li})^2 - c(\lambda_j)(H^{lj})^2) db_s^l}{\lambda_j - \lambda_i},$$

then we have

$$A_t^2 = A_t^4 + \sum_{i<j} \int_0^t \Big[\sum_{k \neq i} \frac{d_s^{ki}}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_i)} - \sum_{k \neq j} \frac{d_s^{kj}}{(\lambda_j - \lambda_k)(\lambda_j - \lambda_i)} + \frac{2d_s^{ij}}{(\lambda_j - \lambda_i)^2} \Big]. \tag{4.17}$$

Since

$$\begin{aligned}
&\sum_{i<j} \Big(\sum_{k \neq i} \frac{d^{ki}}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_i)} - \sum_{k \neq j} \frac{d^{kj}}{(\lambda_j - \lambda_k)(\lambda_j - \lambda_i)} + \frac{2d^{ij}}{(\lambda_j - \lambda_i)^2} \Big) \\
&= \sum_{i<j<k} \frac{(\lambda_k - \lambda_j)d^{jk} - (\lambda_k - \lambda_i)d^{ik} + (\lambda_j - \lambda_i)d^{ji}}{(\lambda_k - \lambda_j)(\lambda_k - \lambda_i)(\lambda_j - \lambda_i)},
\end{aligned}$$

then by the same argument used in [5], with

$$\begin{aligned}
D^{jik} &= \Big[(g^2(\lambda_j) - g^2(\lambda_k))(\sum_l (h^2(\lambda_i)(H^{li})^2 - h^2(\lambda_k)(H^{lk})^2) db^l) \\
&\quad + (g^2(\lambda_i) - g^2(\lambda_k))(\sum_l (h^2(\lambda_j)(H^{lj})^2 - h^2(\lambda_k)(H^{lk})^2) db^l) \Big] \\
&\quad \times (\lambda_i - \lambda_j) \\
&= (d^{ij} - d^{ik} - d^{jk} + d^{kk})(\lambda_i - \lambda_j),
\end{aligned}$$

we obtain that

$$A^2 = A^4 + A^5 + A^6,$$

where

$$A_t^5 = \frac{1}{2} \sum_{i<j<k} \int_0^t \frac{(D_s^{jik} + D_s^{ikj} - D_s^{jki})}{(\lambda_k - \lambda_j)(\lambda_k - \lambda_i)(\lambda_j - \lambda_i)}$$

and

$$A_t^6 = \frac{1}{2} \sum_{i<j<k} \int_0^t \frac{(\lambda_k - \lambda_i)d_s^{jj} - (\lambda_k - \lambda_j)d_s^{ii} - (\lambda_j - \lambda_i)d_s^{kk}}{(\lambda_k - \lambda_j)(\lambda_k - \lambda_i)(\lambda_j - \lambda_i)}.$$

According to the facts that $\sum_l (H^{lj})^2 = 1$ and $\underline{\sigma}^2 t \leq b_t^l \leq \overline{\sigma}^2 t$. We have

$$\begin{aligned}
d_s^{jj} - d_s^{ii} &= g^2(\lambda_j)h^2(\lambda_j) \sum_l (H^{lj})^2 db_s^l - g^2(\lambda_i)h^2(\lambda_i) \sum_l (H^{li})^2 db_s^l \\
&\leq (\overline{\sigma}^2 g^2(\lambda_j)h^2(\lambda_j) - \underline{\sigma}^2 g^2(\lambda_i)h^2(\lambda_i)) ds
\end{aligned}$$

and so

$$
\begin{aligned}
A_t^6 &= \frac{1}{2}\sum_{i<j<k}\int_0^t \frac{1}{\lambda_k-\lambda_i}\Big(\frac{d_s^{jj}-d_s^{ii}}{\lambda_j-\lambda_i}-\frac{d_s^{kk}-d_s^{jj}}{\lambda_k-\lambda_j}\Big)\\
&\le \frac{1}{2}\sum_{i<j<k}\int_0^t \frac{1}{\lambda_k-\lambda_i}\Big(\frac{\overline{\sigma}^2 g^2(\lambda_j)h^2(\lambda_j)-\underline{\sigma}^2 g^2(\lambda_i)h^2(\lambda_i)}{\lambda_j-\lambda_i}\\
&\qquad\qquad -\frac{\underline{\sigma}^2 g^2(\lambda_k)h^2(\lambda_k)-\overline{\sigma}^2 g^2(\lambda_j)h^2(\lambda_j)}{\lambda_k-\lambda_j}\Big)ds.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\left|A_t^6\right| \le \sum_{i<j<k}\int_0^t \frac{1}{\lambda_k-\lambda_i}\Big(&\frac{\left|\overline{\sigma}^2 g^2(\lambda_j)h^2(\lambda_j)-\underline{\sigma}^2 g^2(\lambda_i)h^2(\lambda_i)\right|}{\lambda_j-\lambda_i}\\
&+\frac{\left|\overline{\sigma}^2 g^2(\lambda_j)h^2(\lambda_j)-\underline{\sigma}^2 g^2(\lambda_k)h^2(\lambda_k)\right|}{\lambda_k-\lambda_j}\Big)ds,
\end{aligned}
$$

then thanks to the hypothesis (ii) we obtain

$$\left|A_t^6\right| \le K\sum_{i<j<k} T<\infty.$$

On the other hand, we have

$$
\begin{aligned}
\left|A_t^4\right| &\le \sum_{i<j}\int_0^t \frac{\left|\sum_l (c(\lambda_i)(H^{li})^2-c(\lambda_j)(H^{lj})^2)\right| db_s^l}{\lambda_j-\lambda_i}\\
&\le \overline{\sigma}^2\sum_{i<j}\int_0^t \frac{\left|c(\lambda_i)\sum_l (H^{li})^2-c(\lambda_j)\sum_l (H^{lj})^2\right| ds}{\lambda_j-\lambda_i}\\
&\le \overline{\sigma}^2\sum_{i<j}\int_0^t \frac{|c(\lambda_i)-c(\lambda_j)|\, ds}{\lambda_j-\lambda_i},
\end{aligned}
$$

then

$$\left|A_t^4\right| \le C\frac{p(p-1)}{2}\overline{\sigma}^2 T.$$

Obviously, we have

$$
\begin{aligned}
\int_0^t \frac{D_s^{jik}}{(\lambda_k-\lambda_j)(\lambda_k-\lambda_i)(\lambda_j-\lambda_i)} \le &\int_0^t \frac{(g^2(\lambda_j)-g^2(\lambda_k))(\overline{\sigma}^2 h^2(\lambda_i)-\underline{\sigma}^2 h^2(\lambda_k))}{(\lambda_k-\lambda_j)(\lambda_k-\lambda_i)}ds\\
&+\int_0^t \frac{(g^2(\lambda_i)-g^2(\lambda_k))(\overline{\sigma}^2 h^2(\lambda_j)-\underline{\sigma}^2 h^2(\lambda_k))}{(\lambda_k-\lambda_j)(\lambda_k-\lambda_i)}ds,
\end{aligned}
$$

which implies that

$$\left| \int_0^t \frac{D_s^{jik}}{(\lambda_k - \lambda_j)(\lambda_k - \lambda_i)(\lambda_j - \lambda_i)} \right| \leq 2C^2 T,$$

and so

$$\left| A_t^5 \right| \leq 3C^2 T \sum_{i<j<k} < \infty.$$

Finally, we obtain that $\left| A_t^2 \right| \leq C'T$ and dP is bounded on any interval $[0, T]$. Since M is a classical martingale under each $P \in P$, then by using McKean argument (see [5]) we deduce that $\tau = \infty \;\; P \; a.s.$ for each $P \in P$. The proof is complete. □

5. EXISTENCE AND UNIQUENESS

Faizullah [3], Graczyk and Malecki [5] have discussed and shown by different methods the pathwhise uniqueness of the solutions of stochastic differential equations. We give another result concerning the stochastic matrix differential equations (4.1). To this end, we will need the G-Burkholder–Davis–Gundy inequalities.

Lemma 5.1 (see [4])**.** *Let (B_t) be a real G-Brownian motion. Then we have:*

(i) *if $p \geq 1$, $\eta \in M_G^p([0, T])$ and $0 \leq s \leq t \leq T$, then*

$$\hat{E}\left[\sup_{s \leq u \leq t} \left| \int_s^u \eta_r d \langle B \rangle_r \right|^p \right] \leq C_1 (t-s)^{p-1} \int_s^t \hat{E}\left[|\eta_u|^p\right] du,$$

where $C_1 > 0$ is a constant independent of η,

(ii) *if $p \geq 2$, then*

$$\hat{E}\left[\sup_{s \leq u \leq t} \left| \int_s^u \eta_r dB_r \right|^p \right] \leq C_2 \left| t-s \right|^{\frac{p}{2}-1} \int_s^t \hat{E}\left[|\eta_u|^p\right] du,$$

where $C_2 > 0$ is a constant independent of η.

Theorem 5.2. *Assume that the function $g, h, a, c : \mathbb{R} \to \mathbb{R}$, satisfy the following conditions:*

(i) *Lipschitz condition: For all $X, Y \in \mathbb{R}^{n \times n}$*

$$\left| J(X) - J(Y) \right|^2 \leq A \left| X - Y \right|^2,$$

where $J(X) = g(X)^{ik} h(X)^{jl}$, $a(X)$ and $c(X)$ respectively and A is a positive constant.

(ii) *Linear growth condition: For all* $X \in \mathbb{R}^{n\times n}$

$$|J(X)|^2 \leq K(1+|X|^2),$$

where $J(X) = g(X)^{ik}h(X)^{jl}, a(X)$ *and* $c(X)$ *respectively and* K *is a positive constant. Then the pathwise uniqueness hold for* X_t.

Remark 5.3. A typical example of (i) and (ii) is $g(x) = a(x) = c(x) = x$ and $h(x) = 1$, which corresponds to the SDE

$$dX_t = X_t dB_t + dB_t^T X_t + X_t dt + X_t d\langle B\rangle_t .$$

Remark 5.4. We mean by "the pathwise uniqueness holds" that if X^k, $k = 1,2$ are two solutions of the SDE (4.1) then the equality between the initial values x_k implies that $X_t^1 = X_t^2$ *q.s.* for each $t \in [0,T]$.

Proof. We begin with the proof of the uniqueness. We have

$$dX^{ij} = \sum_{k,l}(g(X)^{ik}h(X)^{lj} + g(X)^{kj}h(X)^{il})dB^{kl} + a(X)^{ij}dt + c(X)^{ij}db^j. \tag{5.1}$$

Let $X(x_k), k = 1,2$ be a solution of the SDE (4.1) with the initial value $x_k = (x_k^{ij})$ and let

$$J^{ijkl}(X) := g(X)^{ik}h(X)^{lj} + g(X)^{kj}h(X)^{il}$$

for $X \in \mathbb{R}^{n\times n}$. Then we have for $u \leq t$

$$\begin{aligned}\left|X_u^{ij}(x_1) - X_u^{ij}(x_2)\right|^2 \leq C\Bigg\{ &\left|x_1^{ij} - x_2^{ij}\right|^2 \\ &+ \sum_{k,l}\left|\int_0^u (J^{ijkl}(X_s(x_1)) - J^{ijkl}(X_s(x_2)))dB_s^{kl}\right|^2 \\ &+ \left|\int_0^u (a(X_s(x_1))^{ij} - a(X_s(x_2))^{ij})ds\right|^2 \\ &+ \left|\int_0^u (c(X_s(x_1))^{ij} - c(X_s(x_2))^{ij})db_s^j\right|^2 \Bigg\}.\end{aligned}$$

By using the BDG type inequalities with $p = 2$ and Lipschitz conditions, we obtain

$$\begin{aligned}&\hat{E}(\sup_{u\leq t}|X_u(x_1) - X_u(x_2)|^2) \\ &\leq C(T,n)\left[|x_1 - x_2|^2 + \int_0^t \hat{E}(\sup_{u\leq s}|X_u(x_1) - X_u(x_2)|^2)ds\right].\end{aligned}$$

We conclude, by using Gronwall's lemma, that

$$\hat{E}(\sup_{u\leq t}|X_u(x_1)-X_u(x_2)|^2)\leq C(T,n)\,|x_1-x_2|^2\,e^{C(T,n)t}.$$

In particular, if $x_1=x_2$ we have the pathwise uniqueness of X_t. For the existence of the solution of (4.1), we consider a Picard sequence ${}^mX=({}^mX^{ij})_{m\in\mathbb{N}}$ defined by:

$$^0X_t^{ij}=x^{ij}\in\mathbb{R},\quad 0\leq t\leq T,$$

$$^{m+1}X_t^{ij}=x^{ij}+\sum_{k,l}\int_0^t J^{ijkl}(^mX_s)dB_s^{kl}+\int_0^t a(^mX_s)^{ij}ds+\int_0^t c(^mX_s)^{ij}db_s^j \tag{5.2}$$

and then

$$\left|^{m+1}X_t^{ij}\right|^2\leq C'\left\{\left|x^{ij}\right|^2+\sum_{k,l}\left|\int_0^t J^{ijkl}(^mX_s)dB_s^{kl}\right|^2\right.$$
$$\left.+\left|\int_0^t a(^mX_s)^{ij}ds\right|^2+\left|\int_0^t c(^mX_s)_s^{ij}db^j\right|^2\right\}.$$

By the BDG type inequalities and linear growth conditions, we have

$$\hat{E}(|^{m+1}X_t^{ij}|^2)\leq C'(T,n)(\left|x^{ij}\right|^2+\int_0^t(1+\hat{E}(\left|^mX_s^{ij}\right|^2))ds),$$

which implies that

$$\hat{E}(\left|^{m+1}X_t\right|^2)\leq C'(T,n)(|x|^2+T+\int_0^t\hat{E}(|^mX_s|^2)ds),$$

and so

$$\hat{E}(\left|^{m+1}X_t\right|^2)\leq C'(T,n)(|x|^2+T)e^{C'(T,n)t}.$$

Now, we will prove that (^mX) is a Cauchy sequence in L_G^2. We have

$$^{k+1}X_t^{ij}-{}^kX_t^{ij}=\sum_{p,l}\int_0^t(J^{ijpl}(^kX_s)-J^{ijpl}(^{k-1}X_s))dB_s^{pl}$$
$$+\int_0^t(a(^kX_s)^{ij}-a(^{k-1}X_s)^{ij})ds$$
$$+\int_0^t(c(^kX_s)^{ij}-c(^{k-1}X_s)^{ij})db_s^j.$$

By an argument similar to the one used in the proof of the uniqueness, we obtain that

$$
\begin{aligned}
\hat{E}(\left|{}^{k+1}X_t - {}^{k}X_t\right|^2) &\le C''(T,n)\Big(\int_0^t \hat{E}(\left|{}^{k}X_s - {}^{k-1}X_s\right|^2)ds\Big) \\
&\le C''(T,n)^2\Big(\int_0^t\int_0^{t_1} \hat{E}(\left|{}^{k-1}X_{t_2} - {}^{k-2}X_{t_2}\right|^2)dt_1dt_2\Big) \\
&\ \ \vdots \\
&\le C''(T,n)^{k+1}\Big(\int_0^t\int_0^{t_1}\cdots\int_0^{t_k} \hat{E}(\left|{}^{1}X_{t_k} - x\right|^2)dt_1\cdots dt_k\Big).
\end{aligned} \tag{5.3}
$$

On the other hand, we have

$$
\begin{aligned}
\hat{E}(\left|{}^{1}X_{t_k} - x\right|^2) &= \hat{E}\Big(\sum_{i,j}\Big(\sum_{p,q} J^{ijpq}(x)B^{pq}_{t_k} + a(x)^{ij}t_k + c(x)^{ij}b^{j}_{t_k}\Big)^2\Big) \\
&\le K(n)\sum_{i,j}\Big[\sum_{p,q}(J^{ijpq}(x))^2\hat{E}(B^{pq}_{t_k})^2 + (a(x)^{ij})^2T^2 \\
&\qquad\qquad + (c(x)^{ij})^2\hat{E}(b^{j}_{t_k})^2\Big] \\
&\le K(n,x,\overline{\sigma},T),
\end{aligned}
$$

which imply that

$$
\left\|{}^{k+1}X_t - {}^{k}X_t\right\|_2^2 = \hat{E}(\left|{}^{k+1}X_t - {}^{k}X_t\right|^2) \le K(n,x,\overline{\sigma},T)\frac{(C''(T,n)T)^{k+1}}{(k+1)!}.
$$

and for each $p, m \in \mathbb{N}$

$$
\begin{aligned}
\left\|{}^{m+p}X_t - {}^{m}X_t\right\|_2 &\le \sum_{k=m}^{m+p-1}\left\|{}^{k+1}X_t - {}^{k}X_t\right\|_2 \le \sum_{k=m}^{\infty}\left\|{}^{k+1}X_t - {}^{k}X_t\right\|_2 \\
&\le \sqrt{K(n,x,\overline{\sigma},T)}\sum_{k=m}^{\infty}\sqrt{\frac{(C''(T,n)T)^{k+1}}{(k+1)!}}\ .
\end{aligned}
$$

Then $({}^{m}X_t)$ is a Cauchy sequence. Let X_t be the limit of ${}^{m}X_t$. In order to complete the proof, we must show that X_t is the solution of the equation (4.1). To this end we have just to prove that

$$\lim_{m\to\infty} \hat{E}\left(\int_0^t (J^{ijpq}(^m X_s) - J^{ijpq}(X_s))dB_s^{pq}\right)^2 = 0,$$

$$\lim_{m\to\infty} \hat{E}\left(\int_0^t (a(^m X_s)^{ij} - a(X_s)^{ij})ds\right)^2 = 0$$

and

$$\lim_{m\to\infty} \hat{E}\left(\int_0^t (c(^m X_s)^{ij} - c(X_s)^{ij})d\left\langle B^{jj}\right\rangle\right)^2 = 0.$$

The first and the third equalities are guaranteed by Lipschitz condition and BDG inequality. For the second inequality, we have by using Holder's inequality and Lipschitz condition

$$\left(\int_0^t (a(^m X_s)^{ij} - a(X_s)^{ij})ds\right)^2 \le TA\int_0^t |^m X_s - X_s|^2\, ds.$$

We conclude by taking G-expectation $\hat{E}$ in both sides and by using the fact that $^m X$ converges to X. $\square$

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